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A REVIEW OF DYNAMIC RESPONSE OF COMPOSITES

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The purpose of this project was to review the state of the art on the
dynamic response of composites and to suggest possible future research
directions. More than three hundred papers are reviewed and compiled at
the end of this report. The existing theories of the dynamic response of
composites are summarized in Chapter I. The harmonic waves and transient
waves in, and the experiments on, composites are discussed in Chapters II,
III and IV, respectively. The differences between various theories and
possible future research directions are presented in Chapter V.

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ABSTRACT

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FOREWORD

A REVIEW OF
DYNAMIC RESPONSE OF COMPOSITES

INTRODUCTION

The dynamic response of a composite has been a hotly pursued research subject since the middle of nineteen sixties. It is a subject which has wide technological applications and also possesses challenging theoretical and experimental problems. Vast literatures are now available on the subject. Various theories have been proposed to predict the dynamic response of a composite. The ultimate goal of research in this area is to obtain an approximate theory which is reasonably simple and, at the same time, is able to predict fairly accurately the response of a composite structure subject to a dynamic loading. Despite the voluminous research papers published, this goal does not seem to have been achieved. This is not a reflection on the lack of research ability in the area. Quite the contrary, there are several sophisticated theories which are able to predict accurately certain aspects of the dynamic response of a composite. This is a reflection on the difficulty of analyzing a composite material. One can have an exact or nearly exact theory which is either mathematically intractable or practically unfeasible. On the other hand, one can have a very simple approximate theory which is too crude to predict even the simplest dynamic response of a composite.

The purpose of this project was to critically review the state of the art on the subject of the dynamic response of composites and to suggest possible future research directions. As we embarked on the project, it soon became clear that the task was a much more difficult one than we

have anticipated. The more than three hundred papers compiled at the end of this report by no means exhaust all papers in the area. Papers which deal with the static response of a composite are not included in the references. A review of the more than three hundred papers reveals that most papers discuss either harmonic waves or transient waves. Of course, there are papers which discuss both harmonic and transient waves. When the material is linear, harmonic waves can be superimposed to obtain a transient wave. This approach is not applicable for nonlinear materials. Therefore, papers which discuss harmonic waves invariably assume that the material is linear.

Published papers on plates and shells which are made of composite materials were also reviewed. Although the review of these papers is not presented here, the papers are included in the References at the end of this report. Likewise, review of papers which model composites as fluids is not presented but they are included in the References.

In Chapter I, we briefly review the existing theories of the dynamic response of a composite. This is followed by a review in Chapter II of papers which deal with harmonic waves in composites. Transient waves in composites are discussed in Chapter III. Although the original objectives of this project do not include reviewing the experimental results, we felt that some experimental results are of sufficient interest and are relevant to the theoretical predictions that a few words should be said about them. This is contained in Chapter IV. Finally, in Chapter V we comment on the differences between various theories and also suggest possible future research directions.

I.   A SUMMARY OF THE EXISTING THEORIES

For most realistic structural composites, an exact description of the static or dynamic behavior is mathematically impracticable. As an alternative, a number of investigators have sought approximate theories. The representatives of such theories are briefly described as follows:

I.1   Effective Modulus Theories

The effective modulus theories such as those proposed by Postma [P6] and White and Angona [W11] replace the actual composite by a homogeneous, generally anisotropic medium whose material constants are a geometrically weighted average of the properties of the constituents. If $C^*_{ijk\ell}$ are such effective moduli of the composite, this theory relates volume averages of stresses to volume averages of strains by a general anisotropic linear stress-strain relation of the form

$$\bar{\tau}_{ij} = C^*_{ijk\ell} \, \bar{\varepsilon}_{k\ell} , \qquad i,j,k,\ell = 1,2,3 \tag{1}$$

Of course, the constants $C^*_{ijk\ell}$ are expressions in terms of the material constants of the constituents and the parameters defining the geometric layout of the composite. The constants $C^*_{ijk\ell}$ satisfy the relations

$$C^*_{ijk\ell} = C^*_{jik\ell} = C^*_{ij\ell k} = C^*_{k\ell ij} \tag{2}$$

Thus, of the 81 constants $C^*_{ijk\ell}$, only 21 are independent. In general, the number of independent elastic constants is much less than 21 because of the existence of symmetries in the structuring of the material. In particular, in the case of a laminated medium consisting of alternating layers of two isotropic elastic materials, the number of independent elastic constants reduces to only 5.

While yielding satisfactory results for certain geometries under static loads, the effective modulus theories exhibit serious deficiencies for virtually all geometries when applied to wave propagation. Specifically, these theories are incapable of reproducing the dispersion and attenuation observed in composites. Such effects become important where dominant signal wave lengths are of the order of the typical composite microdimension. Since dispersion and attenuation are results of the microstructure with discontinuous material properties, any continuum theory must in one way or another take into account the influence of microstructure. The following theories were developed with this purpose in mind.

1.2 Effective Stiffness Theories

The effective stiffness theory was the first continuum model for laminated media and fiber-reinforced composites to account for a typically dynamic effect such as geometric dispersion and hence to reflect the influence of the microstructure of a composite. The theories were developed in a series of papers [S18,A9,G10,G11,A10] by Achenbach, Grot, Herrmann and Sun. Higher order theories of this kind were derived by Turhan [T15].

The theories have been formulated in several different forms, but the case of the linearly elastic laminated composites is perhaps a typical one. Here we outline the theory of elastic waves in laminated composites [S18] briefly. The reinforcing and matrix layers are both assumed to be homogeneous, linear isotropic elastic materials. For elastic waves propagating in the composites, this theory approximates the displacements of the reinforced layer and the matrix layer in the kth cell as

$$u_i^{fk} = u_{oi}^{fk}(x_1, x_2^{fk}, x_3, t) + \bar{x}_2^{f} \psi_{2i}^{fk}(x_1, x_2^{fk}, x_3, t)$$

$$u_i^{mk} = u_{oi}^{mk}(x_1, x_2^{mk}, x_3, t) + \bar{x}_2^{m} \psi_{2i}^{mk}(x_1, x_2^{mk}, x_3, t)$$
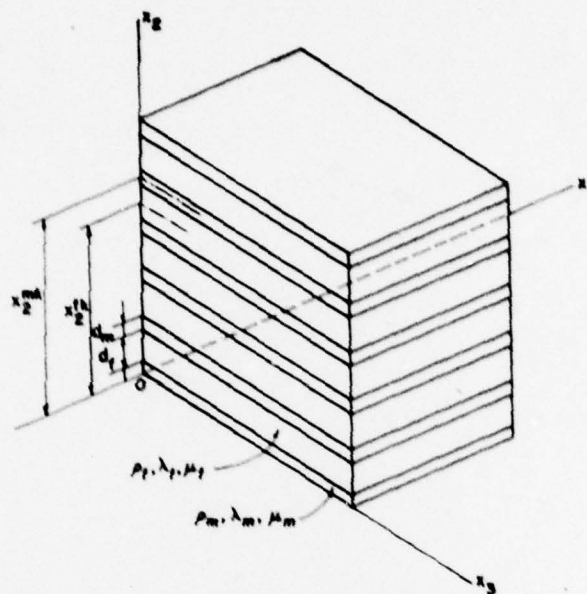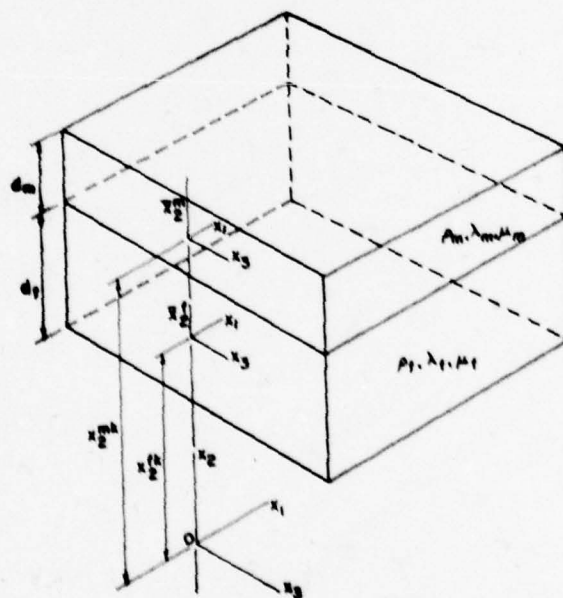
$$(5)$$

Fig. 1 Laminated medium

Fig. 2 Pair of reinforcing and matrix layers

where u_i^{fk} and u_i^{mk} denote the displacements of the reinforcing and matrix layers respectively, u_{oi}^{fk} and u_{oi}^{mk} denote the displacements at the mid-planes of the two corresponding layers, $\bar{x}_2^f$ and $\bar{x}_2^m$ are local coordinates measured from the corresponding midplanes, ψ_{21}^{fk} and ψ_{23}^{fk} represent anti-symmetric thickness shear deformations, and ψ_{22}^{fk} represents symmetric stretch deformation of the kth reinforcing layer. Similar definitions apply for ψ_{21}^{mk}, ψ_{23}^{mk} and ψ_{22}^{mk}. The approximate theory allows dynamic interaction of the layers through the continuity of displacements at the interfaces. This is obtained from Eq. (3) as

$$u_{oi}^{mk}(x_1, x_2^{mk}, x_3, t) - u_{oi}^{fk}(x_1, x_2^{fk}, x_3, t)$$

$$= \frac{1}{2} d_f \, \psi_{2i}^{fk}(x_1, x_2^{fk}, x_3, t) + \frac{1}{2} d_m \, \psi_{2i}^{mk}(x_1, x_2^{mk}, x_3, t) \qquad (4)$$

With the assumption of the displacement fields given by Eq. (3), one can obtain the strain, and consequently the elastic strain energy W^{fk} and W^{mk} in the kth reinforcing layer and the matrix layer, respectively. One also obtains the kinetic energy T^{fk} and T^{mk}. Now, if the composite consists of n reinforcing layers and n matrix layers within a certain thickness ℓ, the total strain energy W_ℓ and kinetic energy T_ℓ are

$$W_\ell = \sum_{\ell}^{n} (W^{fk} + W^{mk}) \qquad (5)$$

$$T_\ell = \sum_{\ell}^{n} (T^{fk} + T^{mk}) \qquad (6)$$

The basic assumption in the effective stiffness theory is the smooth opera-tion in which W_ℓ is expressed in terms of W^f and W^m by

$$W_\ell \cong \int_\ell \frac{1}{d_f + d_m} (W^f + W^m) \, dx_2 \qquad (7)$$

where W^f and W^m are now defined for all x_2. If the layering thicknesses

are small, W^f and W^m are approximately equal to W^{fk} and W^{mk} within each layer. Therefore, the strain energy density W can be defined as

$$W = (W^{fk} + W^{mk})/(d_f + d_m) \tag{8}$$

where W^{fk} and W^{mk} are assumed to hold for all x_2, not just x_2^{fk} and x_2^{mk}. Similar smoothing operation is applied to the kinetic energy. By assuming the smallness in the layer thicknesses, Eq. (4) can be approximated by a differential form:

$$S_i = (d_f + d_m)\partial_2 U_i(x_1, x_2, x_3, t)$$
$$-d_f \psi_{2i}^f(x_1, x_2, x_3, t) - d_m \psi_{2i}^m(x_1, x_2, x_3, t) = 0 \tag{9}$$

Finally, one invokes Hamilton's principle in which the continuity conditions (9) are included by using the Lagrangian multipliers λ_i:

$$\delta \int_{t_1}^{t_2} (T - W - \lambda_1 S_1 - \lambda_2 S_2 - \lambda_3 S_3) dV \, dt = 0 \tag{10}$$

This results in a system of partial differential equations for the displacements u_{oi} and ψ_{2i}.

Sun, Achenbach and Herrmann [S18] then used these displacement equations to study the propagation of plane harmonic waves in a laminated medium. Dispersion relations for harmonic waves propagating parallel to and normal to the direction of the layering were presented, and the approximate dispersion curves were compared with exact curves. The limiting phase velocities at vanishing wave members agree with the exact limits. The lowest antisymmetric mode for waves propagating in the direction of the layering shows the strongest dispersion which is very adequately described by this theory over

a substantial range of wave numbers. Various theories of effective stiffness will be reviewed later.

I.3 Mixture Theories

Another approach, suggested by Lempriere [L7], is to use the mixture theories as models of the dynamics of composites. The fundamental concept of mixture was postulated by Truesdell and Toupin [T12], and further developed by Green and Naghdi [G5,G7,G8], Green and Steel [G6], Steel [S14], and others. In these theories, the constituents of the structural composite are superimposed in space and allowed to undergo individual deformations. The microstructure of the composite is then simulated by specifying the nature of constituent interactions and the form of the mixture constitutive relations.

While general conservation laws governing the mixture may easily be formulated, the practical application to composite materials encounters difficulties in that it is rather difficult to analytically specify the interactions between the constitutents on the basis of the knowledge of the geometry and constitutive relations of the individual constituents. In 1971, Bedford and Stern [B11] first proposed a mixture theory for a laminated composite wherein the interaction parameters were determined on the basis of results of certain simple quasi-static problems. Then, in a series of papers by Bedford and Stern [S15,S16,B13], Hegemier and Nayfeh [H7], and Hegemier, Gurtman and Nayfeh [H8], mixture theories were formulated for certain laminated and fiber-reinforced composites with varying degrees of success. In the following, we outline the binary mixture theory for wave guide-type propagation in laminated and unidirectional fibrous composites formulated by Hegemier, Gurtman and Nayfeh in [H8].

For a periodic array of linearly elastic, isotropic and homogeneous bi-laminates, bonded at their interfaces, these authors first integrated the elastodynamic equations of motion and constitutive relations for each constituent over the thickness of each constituent and defined averaged stresses and displacements over that thickness. By using the condition that σ_{xy} must be continuous across interfaces, they obtained the momentum equations in the form

$$\partial_x \sigma_{xx}^{(1p)} - \rho_1^{(p)} \partial_t^2 u_x^{(1a)} = P$$

$$\partial_x \sigma_{xx}^{(2p)} - \rho_2^{(p)} \partial_t^2 u_2^{(2a)} = - P \tag{11}$$

in which the superscripts or subscripts 1 and 2 refer to 1 and 2 constituents respectively, the superscript a refers to average value, and "partial" stresses and densities are defined as

$$\sigma_{xx}^{(\alpha p)} = n_\alpha \sigma_{xx}^{(\alpha,a)} \ , \qquad \rho_\alpha^{(p)} = n_\alpha \rho_\alpha \ , \qquad \alpha = 1,2 \tag{12}$$

where

$$n_\alpha = h_\alpha / (h_1 + h_2) \ , \qquad \alpha = 1,2 \tag{13}$$

is a volume fraction of the α-constituent, and h_α is one half of the thickness of the α-constituent. In (11), P is an "interaction" term reflecting momentum transfer from one constituent to another via shear interaction across laminate interfaces. By a rational analysis, Hegemier, etc., found the interaction term P taking the form

$$P = \frac{K}{(h_1 + h_2)^2} \left(u_x^{(1a)} - u_x^{(2a)} \right) \tag{14}$$

where

$$K = 3\mu_1\mu_2 / (\mu_1 n_2 + \mu_2 n_1) \tag{15}$$

Similarly, integration of the consitutive relations for the individual constituents followed by a rational analysis, Hegemier, etc., found the constitutive relations for the mixture as follows:

$$\sigma_{xx}^{(1p)} \simeq c_{11}\partial_x u_x^{(1a)} + c_{12}\partial_x u_x^{(2a)}$$

$$\sigma_{xx}^{(2p)} \simeq c_{12}\partial_x u_x^{(1a)} + c_{22}\partial_x u_x^{(2a)} \tag{16}$$

where

$$c_{\alpha\alpha} = \left(n_\alpha E_\alpha - \frac{\lambda_\alpha^2}{E}\right), \quad c_{\alpha\beta} = \frac{\lambda_\alpha \lambda_\beta}{E} \tag{17}$$

$$(\alpha,\beta = 1,2; \ \alpha \neq \beta)$$

in which

$$E_\alpha = (\lambda + 2\mu)_\alpha \tag{18}$$

and λ_α, μ_α are Lamé constants of the α-constituent.

For fibrous composites, Hegemier, etc., approximated a hexagonal array of fibers by concentric, linear elastic cylinders, with perfect interface bonds and subject to vanishing shear stress and radial displacement on the outer boundaries so that for a cylindrical element, $\alpha = 1$ $(r \leq r_1)$ denotes fiber and $\alpha = 2$ $(r_1 \leq r \leq r_2)$ denotes matrix. They found that the momentum equations for fibrous composites can be also written in the form (11) where, for this case,

$$P = \frac{8n_1\mu_1\mu_2}{r_1\left[\frac{4\mu_1 Q}{r_2 - r_1} + r_1\mu_2\right]} (u_x^{(2a)} - u_x^{(1a)}) \tag{19}$$

in which

$$Q = \frac{(r_2 - r_1)^3}{r_1 + r_2} \left(\frac{2}{3}\frac{r_2}{r_2 - r_1} - \frac{1}{4}\right)$$

$$n_1 = \frac{r_1^2}{r_2^2}, \quad n_2 = \frac{r_2^2 - r_1^2}{r_2^2} \tag{20}$$

They also found that the constitutive relations of the mixture still take the form (16) where

$$c_{\alpha\alpha} = (\lambda_\alpha + 2\mu_\alpha^*)n_\alpha - \lambda_\alpha^2/D, \quad c_{\alpha\beta} = \lambda_\alpha\lambda_\beta/D, \quad (\alpha \neq \beta)$$

$$D = \frac{1}{n_1 n_2} [(\lambda_1 + \mu_1)n_2 + (\lambda_2 + \mu_2)n_1 + \mu_2\sqrt{n_1}]$$

(21)

Comparison of exact and approximate phase velocity data for laminated and fibrous composites indicate that the theories just described provide good agreement for wavelengths greater than the typical composite microdimension.

1.4 Continuum Theories Based on Asymptotic Expansions

For laminates and directionally reinforced fibrous composites, considerable success has been achieved in the development of continuum models based upon asymptotic expansion techniques in which the ratio of the characteristic lengths of the structuring to the wavelengths is assumed much smaller than unity. One approach, utilizing direct asymptotic expansions, has been proposed by Ben-Amoz [B22,B23] and is appropriate for problems of the wave-guide type. Another technique, utilizing spatial and asymptotic expansions, has been proposed by Hegemier and developed in a series of papers by Hegemier and Nayfeh [H7], Hegemier, Gurtman and Nayfeh [H8], Hegemier and Bache [H9,B1,H10], and Gurtman, etc. [G13]. The latter applied to problems of both the wave-guide and wave-reflect types. The technique developed by Hegemier, et al., models a heterogeneous composite as a continuum with microstructure. In this theory, the governing equations are completely determined from a knowledge of the geometry and constitutive relations of the composite microcomponents. In addition, this theory provides information on stress and displacement fields within the microcomponents of the composite. A typical example is the elastic waves in laminated composites. We will briefly present the case of wave propagation normal to the laminate in the following.

For an elastic bilaminate, let $h^{(1)}$ and $h^{(2)}$ be the half thickness of the layers and y be the distance perpendicular to the layering. Hegemier, et al. [H7] started from the equations of motion and constitutive relations for the individual layers and expanded the stress and displacement into power series of the local coordinates with origin at the centroid of the constituent layer. By imposing the continuity conditions of the stress and displacement across the layer interface, they obtained the equations in a differential-difference form. Finally, assuming that all difference expressions admit Taylor series expansions in the quantity $\Delta = h^{(1)} + h^{(2)}$, which is the half thickness of a unit cell, they converted the differential-difference equations to partial differential equations. After some algebraic and trigonometric manipulations, the two partial differential equations are shown to satisfy the following global differential equation for Φ and Ψ which are essentially the stress and the strain:

$$\left\{\cosh(2\gamma^{(1)}\epsilon\partial_\tau)\cosh(2\gamma^{(2)}\epsilon\partial_\tau) + \frac{1+\kappa^2}{2\kappa}\sinh(2\gamma^{(1)}\epsilon\partial_\tau)\sinh(2\gamma^{(2)}\epsilon\partial_\tau)\right.$$
$$\left. - \cosh(2\epsilon\partial_\xi)\right\}\left\{\begin{matrix}\Phi\\\Psi\end{matrix}\right\} = 0 \qquad (22)$$

where

$$\xi = y/\ell, \qquad \tau = ct/\ell, \qquad \epsilon = \Delta/\ell \qquad (23)$$

$$\gamma^{(\alpha)} = ch^{(\alpha)}/(c^{(\alpha)}\Delta), \qquad \kappa = \rho^{(2)}c^{(2)}/\rho^{(1)}c^{(1)}$$

In Eq. (23), ℓ is a reference length, $c^{(\alpha)}$ and $\rho^{(\alpha)}$ are the wave speed and mass density of the αth layer, and c is the wave speed of the composite when $\epsilon = 0$. In the limiting case $\epsilon = 0$, Eq. (22) reduces to

$$\left\{\partial_\xi^2 - \left(\gamma^{(1)^2} + \gamma^{(2)^2} + \frac{1+\kappa^2}{\kappa}\gamma^{(1)}\gamma^{(2)}\right)\partial_\tau^2\right\}\left\{\begin{matrix}\Phi\\\Psi\end{matrix}\right\} = 0 \qquad (24)$$

and hence c is obtained by letting the coefficient of ∂_τ^2 to be unity:

$$\gamma^{(1)^2} + \gamma^{(2)^2} + \frac{1+\kappa^2}{\kappa} \gamma^{(1)} \gamma^{(2)} = 1 \qquad (25)$$

Equation (22), when expands in powers of ϵ, can be written as, after making use of Eq. (25),

$$\left\{ (1 + a_2 \epsilon^2 \partial_\xi^2 + a_4 \epsilon^4 \partial_\xi^4 + \ldots) \partial_\xi^2 \right.$$
$$\left. - (1 + b_2 \epsilon^2 \partial_\tau^2 + b_4 \epsilon^4 \partial_\tau^4 + \ldots) \partial_\tau^2 \right\} \left\{ \begin{matrix} \phi \\ \psi \end{matrix} \right\} = 0 \qquad (26)$$

where

$$a_2 = 2^3/4! , \qquad a_4 = 2^5/6!$$

$$b_2 = \frac{2}{3} (\gamma^{(1)^2} + \gamma^{(2)^2}) - \frac{1}{3} (\gamma^{(1)^2} - \gamma^{(2)^2})^2$$

$$b_4 = \frac{4}{45} \left[\gamma^{(1)^2} \gamma^{(2)^2} (\gamma^{(1)^2} + \gamma^{(2)^2}) - (\gamma^{(1)^6} + \gamma^{(2)^6}) \right] \qquad (27)$$

$$+ \frac{2}{15} (\gamma^{(1)^4} + \gamma^{(2)^4}) + \frac{4}{9} \gamma^{(1)^2} \gamma^{(2)^2}$$

This was obtained by Hegemier, et al., [H7]. However, b_4 obtained here is different from that of [H11], and it seems that b_4 of [H11] is in error.

One could have obtained a general expression for the coefficients a_{2n} and b_{2n} of Eq. (27) if one rewrites Eq. (22) in the following form:

$$\left\{ \theta \cosh(2\gamma \epsilon \partial_\tau) - (\theta - 1)\cosh(2\delta \epsilon \partial_\tau) - \cosh(2 \epsilon \partial_\xi) \right\} \left\{ \begin{matrix} \phi \\ \psi \end{matrix} \right\} = 0 \qquad (28)$$

where

$$\gamma = \gamma^{(1)} + \gamma^{(2)}, \qquad \delta = \gamma^{(1)} - \gamma^{(2)}$$
$$\theta = (1 + \kappa)^2/4\kappa \qquad (29)$$

Noticing that Eq. (25) can be written in the form

$$\theta \gamma^2 - (\theta - 1) \delta^2 = 1 \qquad (30)$$

Equation (28) reduces to

$$\left\{ \cosh(2\epsilon\partial_\xi) - \left[\frac{1-\delta^2}{\gamma^2-\delta^2} \cosh(2\epsilon\gamma\partial_\tau) - \frac{1-\gamma^2}{\gamma^2-\delta^2} \cosh(2\epsilon\delta\partial_\tau) \right] \right\} \left\{ \begin{matrix} \phi \\ \psi \end{matrix} \right\} = 0 \qquad (31)$$

Expansion of each term in powers of ϵ leads to

$$\sum_{n=0}^{\infty} \left\{ \left[\frac{2^{2n+1}}{(2n+2)!} \epsilon^{2n} \partial_\xi^{2n} \right] \partial_\xi^2 - \left[\frac{2^{2n+1}}{(2n+2)!} \frac{(1-\delta^2)\gamma^{2(n+1)} - (1-\gamma^2)\delta^{2(n+1)}}{\gamma^2-\delta^2} \epsilon^{2n} \partial_\tau^{2n} \right] \partial_\tau^2 \right\}$$

$$\left\{ \begin{matrix} \phi \\ \psi \end{matrix} \right\} = 0 \qquad (32)$$

Therefore,

$$a_{2n} = \frac{2^{2n+1}}{(2n+2)!}$$

$$b_{2n} = \frac{2^{2n+1}}{(2n+2)!} \frac{(1-\delta^2)\gamma^{2(n+1)} - (1-\gamma^2)\delta^{2(n+1)}}{\gamma^2-\delta^2} \qquad (33)$$

It can be shown that Eq. (33) reproduces Eq. (27) for $n = 1$ and 2.

By letting

$$\left\{ \begin{matrix} \phi \\ \psi \end{matrix} \right\} = \left\{ \begin{matrix} \phi \\ \psi \end{matrix} \right\} \exp\left[ik(\xi - c_p\tau) \right] \qquad (34)$$

where c_p and k are the nondimensional phase velocity and wave number, Eq. (22) reduces to the exact frequency equation obtained by Rytov [R13]. On the other hand, if we substitute Eq. (34) into Eq. (26), then one obtains various approximate frequency equations depending on how many terms in Eq. (26) are retained. If we keep all terms up to ϵ^{2N}, we may call the approximation Nth order theory. Hegemier [H11] used a different definition for the order of approximation. For the Nth order theory, he used Eq. (22) and kept the terms ϵ^{2N} and ϵ^{2N+1} in the power series expansion of cosh( ) and sinh( ). With this definition, the first order theory (N = 1) would include not only the ϵ^2 terms, but also some (not all) terms of ϵ^4 and ϵ^6. Numerical examples show that his first order theory yields better accuracy than several existing theories of the same order.

It should be mentioned that the binary mixture theory of Hegemier, et al., outlined in Section I.3 can be obtained from a modified first order theory of Hegemier.

I.5 Variational Methods

For harmonic waves in a composite with a periodic structure, a variation approach may be employed. This can particularly become a very effective tool, if one uses a variation statement in which not only the displacement, but also the stress field is given independent variation. Moreover, by permitting discontinuity in the displacement and the stress test functions, one can expect a more accurate reproduction of the local variation in the displacement and stress fields within and across the constituent materials. Examples of such calculations can be found in a paper by Kohn, et al. [K5], where the theorem of stationary potential energy which leads to the Rayleigh quotient for the eigen-frequency is used, and in a thesis by Wheeler [W10], and in another thesis by Wu [W17]. Nemat-Nasser [N12] developed more general variation principles in which the displacement, the stress, and the strain in one case, and the displacement and the stress in another case, are given independent variations and which include appropriate general boundary and discontinuity conditions. Here we illustrate Nemat-Nasser's variation principles by using the one-dimensional case as follows.

For waves propagating in an elastic medium whose properties vary periodically in the direction of propagation, i.e., the x-direction, let a be the periodicity-length. Then one has

$$\rho(x+a) = \rho(x)$$

$$\eta(x+a) = \eta(x)$$

(35)

where η stands for $\lambda + 2\mu$ when dilational waves are considered, and for μ when shear waves are considered. Consider harmonic waves propagating normal to the layers of a composite consisting of periodically elastic layers bonded together. Assume that a typical cell in this composite consists of two materials, M^β, $\beta = 1,2$, where M^1 occupies the region $-a/2 \le x \le -b/2$ and $b/2 \le x \le a/2$, and M^2 occupies the region $-b/2 \le x \le b/2$. Now consider the functional

$$J_1 = \int_{-a/2}^{a/2} \left\{ \frac{1}{2} D\sigma\sigma^* + \frac{1}{2} \rho\omega^2 uu^* - \sigma \frac{du^*}{dx} + c.c. \right\} dx$$

$$+ \left\{ \lambda \left[u^*\left(\frac{a}{2}\right) - u^*\left(-\frac{a}{2}\right) e^{-iqa} \right] \right\} + \left\{ \bar{\sigma} \langle u^* \rangle \right\}_{x=\pm b/2} + c.c. \tag{36}$$

in which the superscript star denotes the complex conjugate, the term c.c. stands for the complex conjugate of quantities which precede it, λ is the Lagrangian multiplier, and

$$D = \frac{1}{\eta}, \quad \bar{\sigma} = \alpha\sigma^{(2)} + (1-\alpha)\sigma^{(1)}, \quad \langle u \rangle = u^{(2)} - u^{(1)} \tag{37}$$

where at x_o, $g^{(2)} = g(x_o^+)$ and $g^{(1)} = g(x_o^-)$, g standing for either σ or u in (37), and α is a weighting parameter. The first variation of J_1 is then

$$\delta J_1 = \int_{-a/2}^{a/2} \left\{ \left[D\sigma - \frac{du}{dx} \right] \delta\sigma^* + \left[\frac{d\sigma}{dx} + \rho\omega^2 u \right] \delta u^* + c.c. \right\} dx$$

$$- \left\{ \left[\sigma\left(\frac{a}{2}\right) - \lambda \right] \delta u^*\left(\frac{a}{2}\right) - \left[\sigma\left(-\frac{a}{2}\right) - \lambda e^{-iqa} \right] \delta u^*\left(-\frac{a}{2}\right) \right.$$

$$\left. - \left[u\left(\frac{a}{2}\right) - u\left(-\frac{a}{2}\right) e^{iqa} \right] \delta\lambda^* + c.c. \right\} \tag{38}$$

$$- \left\{ \langle\sigma\rangle \, \delta\bar{u}^* - \langle u\rangle \, \delta\bar{\sigma}^* + c.c. \right\}_{x=\pm b/2}$$

where

$$\bar{u} = (1-\alpha)u^{(2)} + \alpha u^{(1)} , \tag{39}$$

The vanishing of δJ_1 for arbitrary variation of the indicated quantities then guarantees the satisfaction of the field equation, the quasi-periodicity conditions, and the continuity of the displacement and the stress across the two materials within the cell.

Hence, by choosing the appropriate test functions u and σ, in the form of Fourier series for this case, one is able to calculate the frequency and so the Fourier coefficients from J_1, (36), by putting $\delta J_1 = 0$. For this case, numerical results obtained by Nemat-Nasser showed an extremely rapid convergency to the exact results.

I.6 Lattice-type Models

In a fiber-reinforced composite the fibers act as wave guides for a wave propagating in the direction of the fibers. For a wave propagating normal to the direction, the fibers undergo little deformation and essentially act as obstacles interacting with each other and with the surrounding medium in a manner which is similar to the behavior of mass particles in a lattice system. These observations have motivated kinematical assumptions regarding the deformations of the reinforcing elements and of the matrix material which are analogous to those used by the phycists in wave guides and lattice models respectively. Thus, the fibers are considered as long and slender structural elements and the matrix is replaced by a system of springs. A three-dimensional theory of this type was first worked out by Turhan [T15]. A lattice model simulating a periodic structure of laminated plates which are formed by a redistribution of masses and stiffnesses of fibers was first presented by Drumheller and Sutherland [D10]. Related works will be reviewed later on.

I.7 Micromorphic Theory of Continua

A theory of micromorphic continua has been developed by Eringen, et al. [E2,T16], and Habip [H1] in a series of papers. The theory is intended for the prediction of thermodynamic behavior of granular solids, anisotropic and polymeric fluids and, in particular, composite materials. In this theory the mechanical fields are considered as distributions. Partial differential equations governing the moments of fields up to any order have been derived. In this theory, a smoothing operation is also employed in which sums over individual constituents are approximated by integrations over the entire material volume. This theory is, in effect, a non-classical mixture theory. As with the general mixture theories, the general forms for interactions and constitutive relations are postulated. However, the unknown functions and/or constants involved must be determined from experiments.

I.8 The Neighborhood Concept

A neighborhood concept related to the differential-geometric method in the continuum theory of dislocation has been proposed by Ben-Amoz [B21]. In this theory, one avoids the difficulties associated with discontinuous material properties by utilizing a neighborhood averaging technique. Unfortunately, the relations between the displacements of the constituents and the corresponding neighborhood averages must be postulated and/or deduced from experiments. So far this technique has not been further developed.

I.9 Theory of Elasticity with Microstructure

In the continuum models aforementioned the media dealt with all possess a periodic microstructure such as media with equally spaced fibers or periodic laminae. The treatment in such cases is greatly facilitated

by the existing periodicity which enables the derivation of continuum theories based on an analysis of the micromotion in a unit cell. While substantial progress has been made with media possessing a periodic micro-structure, little progress has been made with media lacking periodicity, such as inclusions of arbitrary geometry embedded in a matrix material. For media lacking periodicity, the theory of elasticity with microstructure developed by Mindlin [M4] is certainly an effective tool provided the matrix is an isotropic elastic material. In this theory, a set of equations for the macro-motion that contain in some measure the effects of the micro-motion has been deduced. Recently, Ben-Amoz [B30] has extended the theory of elasticity with microstructure to a heterogeneous medium consisting of inclusions of arbitrary geometry embedded in a matrix material. His treat-ment is structured along the lines of Mindlin's theory although there are important differences. The crucial difference is that the arbitrary material constants in Mindlin's theory are deduced here in terms of known constituent properties. Specifically, two pairs of characteristic constants are identified: both length and time scales associated with dilatational and shear waves. In the following, the dynamic theory for composite materials of Ben-Amoz is outlined briefly. For wave propagation in a heterogeneous medium consisting of inclusions of arbitrary geometry em-bedded in a matrix material, Ben-Amoz has obtained the displacement equations of motion as

$$
\bar{G}_v (1-\beta^2) \nabla^2 u_i + (\bar{\lambda}_v + \bar{G}_v)(1-\beta_o^2) e_{,i} = \left(1 - \frac{1}{32} \gamma^2 \nabla^2\right) u_i - \frac{1}{32} \gamma^2 \ddot{e}_{,i}
$$
$$
+ \frac{1}{4}\left(1 + \frac{1}{16}\nu^2\right)\nabla^2 \ddot{u}_i
$$

(40)

where

$$\lambda_v = \lambda_f v_f + \lambda_m v_m, \qquad G_v = G_f' v_f + G_m v_m$$

$$\bar{\lambda}_v = \lambda_v / (\lambda_v + 2G_v), \qquad \bar{G}_v = G_v / (\lambda_v + 2G_v)$$

$$\beta_o^2 = (\alpha^2 \lambda_v + \beta^2 G_v)/(\lambda_v + 2G_v)$$

$$\alpha^2 = \frac{1}{4} [1 - (v_f - 4I_f)(G_f - G_m)/\lambda_v]$$

$$\beta^2 = \frac{1}{4} [1 - (v_f - 4I_f)(G_f - G_m)/G_v]$$

$$\gamma^2 = \frac{1}{4} [1 + (v_f - 4I_f)(1/\rho_m - 1/\rho_f)\rho_R]$$

$$I_f = \int_{V_f'} r^2 dV_f'$$

$$\rho_R = [v_f \rho_f^{-1} + v_m \rho_m^{-1}]^{-1}$$

(41)

In the foregoing, ρ_f, λ_f and G_f represent the inclusion density and moduli whereas ρ_m, λ_m, G_m represent the corresponding matrix properties; v_f and v_m denote the volume fractions of inclusion and matrix in a unit cell, respectively. The integral I_f taken over the inclusion volume V' in a representative volume V is the polar moment of inertia of the inclusions about the center of the representative volume and thus the effect of inclusion distribution is contained in this integral which enters into the material constants α, β, γ. From (40), Ben-Amoz has extracted the following two systems for rotational and dilatational modes:

$$\bar{G}_v (1 - \beta^2 \nabla^2) \nabla^2 \omega_i = [1 - (32\gamma^2)^{-1} \nabla^2] \ddot{\omega}_i + \frac{1}{4} (1 + \frac{1}{16} \nabla^2) \nabla^2 \ddot{\omega}_i$$

$$(1 - \mu^2 \nabla^2) \nabla^2 e = [1 - (16\gamma^2)^{-1} \nabla^2] \ddot{e} + \frac{1}{4} (1 + \frac{1}{16} \nabla^2) \nabla^2 \ddot{e}$$

(42)

where ω_i are the rotations and e is the dilatation and

$$\mu^2 = \frac{1}{4} \left[1 - \left\{ \frac{(\lambda + 2G)_f - (\lambda + 2G)_m}{\lambda_v + 2G_v} \right\} (v_f - 4I_f) \right]$$

(43)

The systems of equations are valid up to wavelengths of the order of a unit cell dimension and are found to reduce under special assumptions to Mindlin's equations in the long wave approximation. For a harmonic plane wave propagating in the medium, the dispersion curves for a cubic array of spherical particles obtained by this theory are rather similar to the curves sketched in [M4] for the lowest acoustic modes.

I.10 Viscoelastic Analogies

A model for the prediction of the dispersive effects in layered composite materials based upon a viscoelastic analogy has been proposed by Barker [B6]. The model consists of a particular stress relaxing equation of state of the Maxwell type. The parameters involved are defined in terms of the properties of the constituent materials and geometry of the layered composite. The technique, which is semi-empirical, predicts almost exactly the average stress in a unit cell of the laminate. In effect, the model smoothes out the detailed behavior arising from reverberations in the layers of the composite.

I.11 Discrete Continuum Theory

A discrete continuum theory for periodically layered composite materials has been proposed by Chao and Lee [C3]. The treatment is more or less along the line of Achenbach, et al. [A9] but without using the smoothing process. In this theory, the displacement field for each layer is obtained by developing a two-term truncated Taylor series. The governing equations which incorporate interface continuity conditions are derived in the form of a system of differential-difference equations. Application is made to propagation of plane harmonic waves in an unbounded layered medium. Thickness

twist vibrations are studied. Numerical results predicted by this theory agree quite closely with the exact results. In general, agreements are even better than effective stiffness theory, as wave number gradually increases.

I.12 Statistical Approach

McCoy [M2], Bose and Mal [B41,B42] have proposed a statistical approach for longitudinal waves of both compressional and shear types in a fiber-reinforced composite where fibers are randomly distributed but of identical properties. In this theory, the composite is considered to be statistically uniform. The phase velocity and damping of the average waves are obtained by a statistical consideration as functions of the statistical and the mechanical parameters. Correlations in the positions of the fibers is intro-duced. The theory leads to Hashin and Rosen's formulas [H6] for bulk modulus and shear modulus if the correlations are ignored. The correlation terms have a significant effect on the damping property of the composite, especially at high frequencies and concentrations. The effect is to increase the velocity and decrease the specific damping capacity. Ziegler's [Z1] mean wave technique for laminated random media and Krumhansl's [K9] average Fourier-Floquet method for disordered composites are similar to this approach.

I.13 Hydrodynamic Concept

In a series of papers, Tsou and Chou [T13,T14], Torvik [T11], Chou and Wang [C6], Munson and Schuler [M13] developed a theory based on the flow across a selected control volume of the medium to predict the Hugoniot curve of a shock moving in the fiber-reinforced and laminated composites. They derived individual mass, momentum and energy conservation equations. By this

theory, they were able to determine not only the average Hugoniot, but also the integral of the interface shear stress over the width of the shock. In some cases, they obtained good comparisons between theoretical and experimental results on a variety of composite geometries. This hydrodynamic approach seems to be a useful tool for determining the range of response that may be expected under compressive shock loading. Related works will be discussed later.

We have thus summarized most of the published theories in this chapter. We will review published work which used these theories to study harmonic and transient waves in composites in the following two chapters.

II. HARMONIC WAVES

The work on harmonic or sinusoidal wave propagation in composites con-
sists of waveguide analyses, in which the geometrical cross section of the
composite does not vary in the propagation direction, and wave-reflect
analyses, in which the material properties vary periodically in the direction
of propagation. In the waveguide case, for wave propagating in the z
direction, a typical response function f is expressed as

$$f(x,y,z,t) = F(x,y) \exp i(\kappa z - \omega t) \tag{44}$$

and in the wave-reflect case as

$$f(x,y,z,t) = F(x,y,z) \exp i(\kappa z - \omega t) \tag{45}$$

where F is the mode shape, x, y and z are the coordinates, κ is the
wave number, ω the frequency, and t is time. In case of wave-reflect
type propagation, F has the same periodicity as the geometry in the z
direction. The dispersion of the waves is expressed in terms of the rela-
tionship between any two of the quantities c, ω or κ, where $c = \omega/\kappa$ is
the phase velocity. For wave-reflect case, ω is periodic in κ with
period $2\pi/a$, where a is the length of the unit cell in the propagation
direction. These two types of analyses of harmonic waves in composites,
based on various methods, are discussed as follows.

II.1 Exact Theories

The initial study of harmonic or sinusoidal waves in laminates was
made by Rytov [R13]. By use of elasticity theory, he obtained the exact
solutions for the case of dilatational waves propagating normal to the
laminates (wave reflection problem) and the case of symmetric waves
propagating parallel to the layers (waveguide problem). He presented the

phase velocity spectrum for each case. The zero frequency limit of the primary mode, which corresponds to the static elastic solution, was also obtained. Rytov's exact solution is widely used as a basis for estimating the accuracy of the theories of continuum for composites.

Sun, Achenbach and Herrmann [S20], and Achenbach [A11] discussed the time-harmonic waves in layered composite materials propagating in the direction of the layering. They considered a medium of alternating layers of two different homogeneous materials. Using the solutions of the equations of elasticity representing plane time-harmonic waves, they derived exact dispersion relations for both cases of symmetric deformations and antisymmetric deformations. The results obtained show that for the high values of the ratio of the shear moduli, or the layering stiffness, the dispersion curves depart sharply from the limiting phase velocities for "long waves" at very small wave numbers. Thus, they concluded that the applicability of the effective modulus theory for wave propagation in practical laminates is very limited since it cannot account for dispersion.

Puppo, Feng and Haener [P7] analyzed sinusoidal wave propagation parallel to fiber direction in a unidirectional fiber-reinforced composite. They modeled the hexagonal array of circular fibers in a matrix by a circular fiber with concentric cylinder of matrix. They then solved the concentric rod problem by the method of elasticity. It appears from their numerical results that the calculation of the parameter α, which is the coefficient of the second term in the expression of the phase velocity in terms of wave number was somewhat questionable.

Hoffman, et al. [H17] also did the analysis of the concentric rod, which models the hexagonal array of circular fibers in a matrix material, and compared it with the results for the equivalent laminate for glass-epoxy

constituents. A very similar qualitative nature may be seen but the parameter α for the two geometries is unequal.

Reuter [R7] studied the dispersion of flexural waves in circular bimaterial cylinders. He obtained the general form for displacements and the frequency equation for the first mode of flexural wave propagation in an infinitely long circular bimaterial cylinder. The theory follows the technique developed by Pochhammer and Chree for elastic bars which is well known in the theory of elasticity. They presented several first-branch dispersion curves for the first flexural mode for various ratios of the constituent cylinder radii. Dispersion characteristics significantly different from those predicted by the theory for homogeneous cylinders are realized.

Lai, Dowell and Tauchert [L1] also provided a thorough treatment of propagation of harmonic waves in a composite elastic cylinder. They showed rigorously how special conditions corresponding to particular combination of material properties can be derived from the general solution. The problem treated in detail pertains to a composite circular elastic rod of infinite length consisting of two layers. The central portion is solid while the outer portion is a cylindrical shell. Their numerical results, in terms of frequency and real wave number, were given for a composite rod of a soft core with a stiff casing. The results were checked with the asymptotic frequency equations at short wavelength. The exploration of the dispersive phenomena and the development of various simplified theories for harmonic waves in a composite rod can be based on the results of this investigation.

Bade, et al. [B2] considered the Bloch-type wave propagation in a three dimensionally fiber-reinforced composite of orthogonal type. The fibers in the direction of propagation (the z-direction) were assumed as a round fiber surrounded by a circular sheath consisting of material for which the properties were obtained by homogenizing the resin matrix and the lateral fibers running in the two directions orthogonal to propagation. They performed the Bloch analysis by expanding the displacements in a three-dimensional Fourier series and deriving an infinite-order matrix for the coefficients of the Fourier series. The determinant of this matrix provides the dispersion relation for the sinusoidal waves. The infinite matrix was solved in two ways: (1) a perturbation technique and (2) a truncation technique in which only a finite number of terms in each of the series was used. It was found that the layering in the direction of propagation has little effect on the dispersion. Calculations using only three terms in the Fourier series in the lateral directions and just the zeroth (averaging) term in the propagation direction were carried out for both isotropic and orthotropic fiber-bundle properties. The isotropic results agree well with the known axisymmetric waveguide solution while the orthotropic properties give much better agreement with experiment.

Sve [S42] carried out an exact analysis of time-harmonic waves traveling obliquely in a periodically laminated medium. The analysis was based on two-dimensional equations of elasticity and Bloch theory. Dispersion relation was obtained for harmonic waves propagating in an arbitrary direction. Limiting phase velocities were presented for infinite wavelength for any angle of propagation in the form of a fourth-order determinant. In case of propagation along or across the layers, this determinant reduces to two determinants of second order that yield the limiting phase velocities

directly. His numerical results indicate clearly the dependence of dispersion upon the angle of propagation.

Sve [S43] also investigated thermoelastic waves in a periodically laminated medium. By an exact analysis, he studied the effect of thermoelastic coupling on sinusoidal waves propagating in the directions parallel, perpendicular and oblique to the planes of laminates. The effect of thermoelasticity is to cause complex propagation constants to occur except in the case of shear wave perpendicular to laminates in which the response is unaffected. His numerical results indicate that thermoelastic attenuation is confined primarily to the quasi-longitudinal modes. The phase velocities and mode shapes are also influenced by the attenuation parameters, especially for large frequencies.

Christensen [C9] presented an analytical formulation of the effective attenuation of harmonic waves, of low frequency, through layered elastic medium. The effective attenuation is defined as the difference in transmitted energy to initial total energy. This energy is accounted for through secondary wave scattering effects resulting in pulse dispersion. When the layers have equal impedance and equal stiffness, pulse attenuation vanishes. As a result of the analysis, an explicit expression for attenuation was derived by a perturbation technique. The theory presents a useful analytical result in wave propagation in laminated composites.

Lee and Yang [L4], and Lee [L3] analyzed the harmonic waves in composite materials with periodic structure of elastic constants and density variation. They employed Bloch or Floquet theory and treated the propagation in terms of Floquet waves. The theory was presented for a laminated composite material and propagation normal to the lamination. They found that the frequency spectrum has a banded structure, comprising pass or propagating bands and

stop bands. It was shown that the frequencies at the boundaries of the bands correspond to wave profiles which are normal modes of vibration of the individual cells with fixed or free surfaces. Both types occur at each limiting frequency. They also interpreted properties of Floquet waves in terms of normal mode theory and interpreted the high frequency limit for Floquet waves in terms of geometrical optics type analysis.

Schoenberg [S1] considered plane sinusoidal waves propagating through a medium made up of plane layers of anisotropic homogeneous linearly elastic material. Using a matrix formulation, he found the stresses and displacements in terms of the boundary conditions on one boundary, $z = 0$. He calculated the generalized transfer function $\bar{\bar{R}}$ explicitly. The transfer function $\bar{\bar{R}}$ can be thought of as a transfer function between the solutions at $z = 0$ and $z = z_n$, where n is the layer number. $\bar{\bar{R}}$ is a function of frequency ω and the material parameters in each of the n layers. In the special case of normal incidence, he found that the eigenvalues are solutions to a bicubic equation instead of a sextic equation.

Sutherland [S39] analytically predicted phase velocity and attenuation for two specific composites using the time-temperature superposition principle to yield the viscoelastic portion of the total dispersion spectrum. The two composites, he considered, are quartz cloth embedded in a phenolic matrix and stainless steel reinforced epoxy. He speculated that the difference between the total spectrum and the viscoelastic portion is then ascribed to the effects of internal geometry. He determined the total spectrum experimentally but not analytically. The results are applicable to harmonic waves traveling perpendicular to the fiber direction once the corresponding total spectrum has been determined.

Kaul and Herrmann [K1] considered free vibration of an elastic cylinder with laminated periodic structure. The circular cylinder considered has a periodic variation of elastic constants and density normal to the axis of the cylinder. They then developed the theory of torsional vibrations of such a cylinder in terms of Floquet or Bloch waves which are quasi-periodic waves and whose amplitude profile has the same periodicity as that of the material and repeats with the periodicity of the cell. Using Floquet's theory, they obtained the dispersion spectrum for time-harmonic waves propagating in such a periodically laminated cylinder. It was shown that the dispersion spectrum has a band structure, consisting of passing bands and stopping bands. Motion in the case of grazing incidence, and motion at the end of the zones were discussed. It was also shown that as the radius of the cylinder tends to infinity, the torsional waves in a circular cylinder degenerate to SH-waves in laminated plates.

II.2 Variational Approaches

Kohn, Krumhansl and Lee [K5] employed variational methods based on the Floquet or Bloch theory to study the propagation of harmonic elastic waves through composite materials of periodic structure. In these methods, variational principles were developed in the form of integrals over a single cell of the composite. The variational principles provided a means of determining phase velocities and stress distributions in Floquet waves traveling through the composite unchanged in form from cell to cell. Strain energy principle was used and possible jump conditions were discussed. The Rayleigh-Ritz procedure was applied to the solution of the variational equation to calculate dispersion relations, the phase velocities and stress profiles. The one-dimensionally periodic system was evaluated and compared with the exact

solution. The results show that this approach, by using smooth displacement test functions, provides a satisfactory determination of natural frequencies and phase velocities, but is inadequate for stress profile. In principle, this method can be used to study time-harmonic waves propagating in composites with rather general structuring as long as the composites are periodic from cell to cell.

Bevilacqua, Krumhansl and Lee [B35] made a generalization of the previous work based on strain energy consideration. The Rayleigh-Ritz procedure was adopted in which the displacement fields were expressed by complex Fourier series in filament and by simple Fourier series in matrix. Consequently, they obtained stress profiles more accurately than those when the whole displacement field was expressed by simple Fourier series in the previous work.

Bevilacqua and Lee [B36] utilized different polynomial representations in filament and matrix for a simple model composite problem comprising a slab of each material with fixed boundary conditions. This simple model provided the same difficulty of a discontinuity in strain at interface, but provided a simpler case for assessing various approaches. It was pointed out that Fourier series representation is less convenient in two-dimensional and three-dimensional cases when the polynomial form may prove to be superior.

Tobón [T10] also made an analysis of propagation of elastic waves in composite materials by variational methods. He carried out a minimum strain energy calculation using a Fourier series expression for the displacement field. In the one-dimensional case, this method of evaluation is unsatisfactory for stress profiles as shown in [K5]. It seems that successful two-dimensional and three-dimensional calculations by this method is feasible if discontinuities are permitted in strain energy calculations.

Bevilacqua and Lee [B37] presented another variational statement based on complementary energy consideration so that the variational integral is expressed in terms of stress. Starting with a continuous displacement variation in Fourier series form, they first determined the corresponding stress variation by integrating the equation of motion. Then inserting this result into the variation integral, they obtained a matrix eigenvalue problem to determine the values of fundamental frequency. The corresponding stress profile obtained exhibits the correct form of stress-gradient discontinuity at the interfaces, and provides an accurate approximation. However, the frequency is less accurately predicted than by their improved strain energy approach [B35].

Variational methods were also explored by Wheeler and Mura [W10], and Wu [W17]. Instead of using the procedure of Rayleigh-Ritz, Wu employed, however, the Galerkin procedure.

In a series of papers [N12-16], Nemat-Nasser applied more general variational principles to the Floquet wave problem than those considered by Kohn, Lee, et al. Based on Hellinger-Reissner variational method, he developed general variational principles in which the displacements, the stresses and the strains in one case, and the displacements and the stresses in another case, are given independent variations, and which include appropriate general boundary and discontinuity conditions. From the general variation principle, he then derived a new quotient, in contrast to Rayleigh quotient, to determine the frequencies of harmonic waves in composite. Waves propagating normal to the layering in a laminated medium were analyzed in detail, and dispersion curves were presented. Numerical results show an extremely rapid convergency to the exact results and are more accurate than those obtained by Kohn, et al. Also, based on the new quotient, he developed quite accurate lower and upper bounds for frequencies of the harmonic waves.

Nemat-Nasser, Fu and Minagawa [N17] further investigated harmonic waves in one-, two- and three-dimensional composites using a new quotient derivable from a general variational method in which both stresses and displacements are varied independently. The general energy method was formulated by assuming periodic structure with continuous and continuously differentiable mass density and elastic constants within the unit cell. The final result is in a simple form and represents either upper or lower bounds on the frequencies. It was shown that the new quotient yields upper bounds if the stress variables are free, and lower bounds if the displacment varia- bles are free. Procedures on error estimating were also given. Numerical results show the superiority of the new quotient compared to the Rayleigh quotient used by Kohn, et al.

Nemat-Nasser and Minagawa [N18] again compared Rayleigh quotient with the proposed new quotient to examine reasons for the astonishing accuracy of the latter. Comparison led to a scheme for obtaining improved test func- tions which give very accurate bounds for frequencies. The scheme seems to be extremely effective in all cases of harmonic waves in layered composites.

11.3 Effective Modulus Theories

Behrens [B18] treated the propagation of elastic waves of Bloch form in a lamellar periodic composite for long wavelength cases. The analysis yields expressions for the five independent effective elastic constants, which he called averaged elastic constants, of the material. The theory is valid when the wavelength is long compared with the intercomponent spacings of the composite and when the composite is a periodic structure.

Behrens [B19] also analyzed the propagation of elastic waves of Bloch form in a filamentary composite for long wavelength cases. The phase velocity and the nine "homogenized" elastic constants were obtained. Again,

the theory is valid when the wavelength is long compared with the intercomponent spacings of the composite.

Behrens [B20] again discussed long waves of Bloch form traveling in filamentary composites. He showed that such stable Floquet waves or Bloch waves retain their form relative to the periodic structure of the composite as they travel. He treated the problem using the "method of long waves" which he proposed previously. What he actually treated is a cylindrical inclusion in a cylindrical medium, a very idealized condition for typical fiber composites.

Using an approach similar to that used in studying the propagation of electromagnetic waves through random media, Mok [M5] and Osten [O2] discussed the sinusoidal dispersion in a fiber-reinforced material. They considered isolated, transverse, cylindrical and/or spherical inclusions and then combined them by random averaging to obtain sinusoidal dispersion solution for composites. The effective modulus and density were then obtained from the dispersion solutions. Both of them are complex numbers and depend on wave frequency. This fact apparently indicates the existence of dissipation and dispersion in the composite under dynamic loadings.

Weitsman [W7] presented an analytic treatment of wave propagation in unidirectional continuous filament composite material. The theory is based on inextensible fiber reinforcement and therefore should be applicable to unidirectional composites reinforced with closely spaced fibers which are much stiffer than the matrix material. The theory is also restricted to waves of sufficient length that geometric dispersion is not significant. He showed the existence of three types of harmonic waves and discussed their associated characteristics such as slowness surfaces, velocities, propagation directions, displacements and energy flux. He also considered wave scattering

at a plane of fiber misalignment. In general, when any of the three wave types impinge on such a plane, reflected and transmitted waves of all three types are generated. Condition under which Rayleigh surface wave may exist was also discussed.

Weitsman [WS] further investigated the reflection of harmonic waves in fiber-reinforced materials. Two cases were considered: (1) fibers are inextensible, (2) fibers are almost inextensible. He showed that three types of harmonic waves exist for each case. The reflections of each type of wave from a plane free surface-bounding a half-space were investigated, and numerical results were presented. Differences and similarities between the nature of wave propagation of the two cases were discussed. The results might be valuable for composite material that are sparsely reinforced by highly rigid fibers although the practical significance of the theory is limited by the simplifying assumptions incorporated into the mathematical model.

Datta [D1] considered a unidirectional fiber-reinforced composite through which elastic shear waves propagate in the direction perpendicular and polarization parallel to fiber direction. Fibers were assumed to be randomly distributed and of elliptic cross section with transverse axes either aligned or randomly oriented. Wave equation was solved in the limit of long wavelengths making use of results for single fiber in infinite medium. Since multiple scattering was neglected, solution is limited to small fiber-volume ratios. For perfect orientation of fiber axes, he derived two effective transverse shear moduli. A limit of narrow cavities with vanishing volume (Griffith cracks) was discussed. It is useful in fracture mechanics of composites.

Weitsman and Benveniste [W9] consider the propagation of harmonic waves in composite material reinforced by fibers whose directions vary continuous in space. They focused special attention on the case of slight variations in fiber directions and on plane waves that propagate parallel to the unperturbed direction; a case in which the solution can be expressed completely in analytical form. They concluded with a solution to a specific example in which they assumed that the fiber directions vary linearly with the angle from an initial to a final position. In a discussion of the specific solution, they pointed out that this linear variation in fiber directions gives rise to secondary waves throughout the composite body; and these results tend to indicate that it may be possible to detect internal deviations in the fiber directions by means of dynamic tests. The basic ideas presented in their work follow that classical formulation of wave propagation in anisotropic materials. The fiber-reinforced composite was represented by means of an equivalent transversely isotropic medium. The solution is valid for long wavelengths.

Shoenberg and Weitsman [S11] further studied plane harmonic waves propagating in composite reinforced by fibers with periodically varying directions. The fibers were assumed to wobble periodically about a dominant direction and all fibers were assumed to be parallel to each other. The fiber-reinforced composite was represented by an equivalent, transversely isotropic medium whose preferred direction coincides with the direction of the fibers. The wobbliness endows the material with a structural periodicity which generates dispersion at all frequencies and instability for various frequency bands. The zones of instability were analyzed by the perturbation method. Bounded solutions and unbounded solutions were discussed. The existence of unbounded solution is equivalent

to the "parametric resonance" encountered in other dynamic problems. The analysis is valid only for wavelengths which are larger than by an order of magnitude the cell dimension of the composite.

II.4 Effective Stiffness Theories

As mentioned earlier, the effective stiffness theory was the first continuum model for laminated media and fiber-reinforced composites to account for geometric dispersion. For a fiber-reinforced composite a simple form of the theory was developed by Achenbach and Herrmann [H14,A11]. They employed the theory to study the propagation of plane harmonic waves in the directions of the fibers and normal to the fiber directions. They found that plane transverse waves propagating in the direction of the fibers are dispersive. The dispersion curves for boron epoxy and glass epoxy depart at small wave numbers from the constant phase velocity predicted by effective modulus theory. It was also found that the dispersion of longitudinal waves for small wave number is negligible as compared to dispersion of transverse waves. It appears that in order to account for the dispersion of longitudinal waves propagating in the direction of the fibers, more complex model is needed.

To improve the simple model for fiber-reinforced composites, Achenbach and Sun [A18] constructed a more accurate homogeneous continuum theory of the effective stiffness type for a fiber-reinforced composite. They considered a fiber-reinforced composite of unidirectional fibers in a rectangular array embedded in a matrix material. The model was based on expansions of the displacements across representative cells. The transition from the actually inhomogeneous composite to a homogeneous continuum was achieved by introducing continuum fields for gross displacement and local deformations. The effective

stiffness theory for fiber-reinforced composites was then derived. For the transverse harmonic waves propagating in the direction of the fibers, phase velocities were calculated numerically. The results show that for wavelengths that are of the order of magnitude of a fiber-diameter or a distance between fibers, the phase velocity is markedly dependent on the wavelength if the elastic constants of the reinforcing material differ substantially from those of the matrix material.

An analogous effective stiffness theory for fiber-reinforced composites with rectangularly-spaced rectangular fibers was worked out by Bartholomew and Torvik [B9], and was applied to discuss the harmonic waves propagating in such composites. Expressions for dispersion of longitudinal and flexural waves propagating parallel to the fibers were derived. Propagation across the fibers was also briefly considered. Again, a very marked dispersion was observed when the elastic constants of the fibers differ substantially from those of the matrix material.

For a laminated medium, Sun, Achenbach and Herrmann [S18,S17,A8,H14, A12] derived an effective stiffness theory in terms of macroscopic displacements plus additional independent variables which give micromechanical variation in the unit cells. They employed the governing equations to determine curves relating the phase velocity to the wave number for harmonic waves propagating parallel to and normal to the layering, and they compared these results with the exact solutions. The lowest antisymmetric modes for waves propagating in the direction of the layering show the strongest variation of the phase velocity over a substantial range of wave numbers. The limiting phase velocities at vanishing wave numbers agree with those of the effective modulus theory and with the exact limits. It was pointed out that their theories of the type bear close resemblance to Mindlin's theory of linear elasticity with microstructure.

Achenbach, Sun and Herrmann [A9] further developed the effective
stiffness theory. The constitutive equations, the stress equations of
motion, and natural boundary conditions were presented, and sufficient
conditions for a unique solution were discussed. They employed the
governing equations and boundary conditions to study the thickness-twist
motion of a laminated composite. For every eigenvalue there is a low
frequency acoustic mode and a high frequency optical mode. The frequencies
of the acoustic modes were compared with the corresponding frequencies pre-
dicted by the effective modulus theory, and the relative magnitudes of the
material parameters for which these frequencies are substantially at variance
were indicated.

The governing equations of the effective stiffness theory were employed
by Sun [S21] to study surface waves propagating along the free surface of a
layered half-space. Several examples of time-harmonic surface wave motion
were considered including waves in a laminated layer over a rigid half-space,
Rayleigh waves and Love waves in a laminated layer over an elastic half-space.
Dispersion curves were presented and comparison was made to results from the
effective modulus theory and to exact solutions where these are available.
These show that the dispersion curves are in good agreement with the exact
solutions and that the effective stiffness theory gives a better approximation
than the effective modulus theory--an advantage gained at the expense of a
more complicated procedure than that of the effective modulus theory.

The effective stiffness theory was refined to include second-order
terms in the displacement expansions by Achenbach and Herrmann [A13]. Time-
harmonic wave propagation in laminated medium (unbounded) was then studied
by means of this refined theory. The exact and approximate dispersion curves
show good agreement. Also, time-harmonic waves propagating in laminated

plate was analyzed by employing this refined theory. Comparison of the
dispersion curve for flexural motion with an exact curve shows similar
qualitative behavior. The qualitative difference is due to "material dis-
persion" which is thus seen to be significant for smaller wavelengths even
in the presence of "geometrical dispersion."

The propagation of time-harmonic waves traveling in an arbitrary
direction in a periodically laminated medium was examined by Sve [S42]. In
addition to an exact analysis, he applied the effective stiffness theory to
the problem and obtained the corresponding approximate dispersion relation.
Numerical results also indicate the dependence of dispersion upon the angle
of propagation. A comparison with the exact solution shows that agreement
is satisfactory for those angles where the dispersion is the strongest.

Higher-order refinements of the effective stiffness theory were
developed by Drumheller and Bedford [D11] for a laminated medium. They
presented improved displacement and stress interface boundary conditions
suitable for higher-order (higher than first order) theories so that the
resulting theory is in a form suitable for the solution of dynamic process
including determination of stresses. Two examples of time-harmonic wave
propagation show that the inclusion of stress boundary condition has
negligible effect on the dispersion curves while the mode shapes are
substantially modified. These indicate that an effective stiffness theory
adequate for the determination of stresses must include stress interface
boundary conditions. Consequently, they obtained more accurate dispersion
curves.

The effective stiffness theories as described above are based on the
construction of strain and kinetic energy densities, and the subsequent use
of Hamilton's principle. For constituent materials that are not perfectly
elastic this approach encounters difficulties. An alternative way of deriving

the balance equations of linear momentum and moment of momentum directly from physical arguments based on momentum considerations was presented by Achenbach [A16]. This alternative method has the advantage that it can be used for composites in which the individual layers are not elastic.

Grot and Achenbach [G10] extended the alternative formulation of effective stiffness theory to a viscoelastic laminated composite including temperature effects. They derived a set of constitutive equations for a laminated medium composed of layers of two anisotropic thermoviscoelastic solids. The special cases of isotropic thermoviscoelastic layers, anisotropic thermoelastic layers and isotropic thermoelastic layers were briefly discussed. This nonlinear theory has the advantage that it can be used for composites which are not elastic.

An extension of the effective stiffness theory to include large deformations and nonlinear material behavior was also carried out by Grot and Achenbach [G11] for a laminated elastic composite. The resulting system of nonlinear field equations, consisting of balance equations, constitutive equations and constraint conditions, bear a close resemblance to equations defining a nonlinear theory of elasticity with microstructure. The equations were employed to study the propagation of small amplitude time-harmonic waves superimposed on a large static deformation. The results are very similar to those in linear theory of effective stiffness except that the coefficients that appeared in the dispersion relation depend not only on the structuring but also on the large static deformation.

An effective stiffness theory for a composite of a cylindrical geometry was developed by Chou and Achenbach [C7]. They presented field equations governing the mechanical behavior of layered cylinders. Grot [G12] proposed an effective stiffness continuum model for curvilinear laminated composites

of a very general type. The model was developed by expanding the variables in a series within each layer and averaging over the thickness. A constitutive theory for nonlinear elastic materials was then developed, and special cases of the theory (linear cases) for cylindrical and spherical laminae were presented and discussed. It was expected that if the ratio of the thickness of the layering to the radius of curvature of the layering is small, the approximate theory with two term expansion in describing the behavior of the laminated body should be as adequate as the approximate theories of plane laminates presented earlier.

Drumheller and Bedford [D12] used a modification of the simpler micro-structure theory developed earlier for elastic laminates by Sun, Achenbach and Herrmann [S18] to analyze plane harmonic wave propagation in elastic laminates. This new version incorporates higher-order thickness variations in the displacement functions and includes restrictions on both displacement and stress at the laminate interfaces. To assess the potential of the second-order microstructure theory for accurate modeling of mechanical processes in laminates, dispersion curves and especially mode shape data for both displacements and stresses were obtained and compared to corresponding exact solutions. The comparisons indicate that while dispersion curves may be nearly identical, extremely significant differences may be observed in the mode shape. It provides a strong evidence that dispersion comparison alone cannot constitute a valid criterion for the assessment of a microstructure theory.

Hlavacek [H15] used an effective stiffness theory for fiber-reinforced composites with hexagonal packing of unidirectional fibers. In the theory a typical hexagonal fiber and matrix layout is replaced by circular cylinders

possessing the same volume of materials. He then illustrated the essence of this theory by plane harmonic waves. Propagation in directions parallel and normal to the fibers were considered. A correspondence between the elastic constants of an equivalent transversely isotropic medium and the parameters of this effective stiffness model was shown using various wave speeds at vanishing wave numbers.

Hlavacek [H16] also presented an effective stiffness model for isotropic two phase elastic composites containing spherical inclusions. His model consists of two concentric spheres, the inner one represents an inclusion, the outer one the matrix. He assumed a continuous field of gross displace- ments u_i which was assigned to the interior of the smaller sphere and to the outer surface of the outer sphere, using a linear expansion about the common center. In addition, he assumed a local homogeneous deformation of the matrix described by a continuous tensor field. The local displacements in the matrix were then fixed by the condition of continuity at the inter- face, and by assuming a linear dependence on the radius. In this way, he arrived at an elastic potential density as an expression in terms of the local deformation and the gradient of the gross displacements. Dispersion curves for harmonic plane waves were presented. The limit phase velocities for vanishing wave numbers yielded the effective moduli which, for a tungsten- carbide-cobalt alloy, agree very well with Hashin and Shtrikman's calculations [H5].

II.5 Interacting Continuum Theories and Theories of Mixtures

Bedford and Stern [B11] proposed a one-dimensional mixture theory for a layered medium wherein the interaction parameters were determined on the basis of results of certain simple quasi-static problems. The novel feature

of their theory is that each constituent is permitted to have an individual
motion, with mechanical coupling between the motions introduced to model
constituent interactions in the actual composite. They employed this theory
to study the time-harmonic waves propagating parallel to the interfaces of a
laminated composite. The dispersion curve so obtained shows good agreement
with the exact solutions.

Stern and Bedford [S16] developed a three-dimensional model based upon
the mixture theory. Results were applied to plane harmonic waves propagating
in a layered medium. Dispersion characteristics based on this theory were
compared with "effective stiffenss" theory and exact solutions. It was found
that neither the mixture theory nor the "effective stiffness" theory could
describe adequately the material response over a wide range of wavelengths.
Only for disturbances propagating in the direction of the layering, this
theory reflects fairly accurately the proper kind of dispersive behavior
and hence may be regarded as a good model for such analyses.

Bedford and Stern [B13] proposed a multi-continuum mixture theory
based on the development of the equations by means of two coupled
media in which the fiber directions are natural axes of symmetry for the
fiber-reinforced material and the transverse isotropy also corresponds to
laminated material where the directions normal to the layers are the natural
symmetry axes. The results are the basic equations for the considered media.
Martin [M1] used this mixture theory to predict the phase velocity of plane
waves in a fiber-reinforced elastic solid. He found the phase velocity as
a function of the wave number and as a function of the direction of propaga-
tion relative to the fiber direction. Also, the particle motion of each of
the constituents was solved for at various wave numbers. It was found that
there are four possible propagation velocities in a binary mixture of this

type, and that two of these waves correspond to nearly shear motion while the remaining two correspond to nearly compressive motion.

Hegemier and Nayfeh [H7] developed a continuum theory for wave propagation normal to the layering of laminated composites based on an asymptotic scheme. The hierachy of models are defined by the order of truncation of the asymptotic sequence. Based on the calculation of the lowest-order dispersive model, accuracy superior to several existing theories was observed. They then cast the lowest-order theory in a standard mixture form. For harmonic waves propagating normal to the laminates, the phase velocity spectra of this mixture theory exhibit good correlation with exact results for the first mode to the cutoff frequency. The simplified theory in binary mixture form has practical applications in view of its simplicity.

Hegemier, Gurtman, Nayfeh and Bache [H8,H9] discussed the waveguide type propagation both in layered and fiber-reinforced composites. Based upon asymptotic expansions, again they developed a simplified first-order mixture theory of which the momentum interaction term is identical in form with Bedford's but the constitutive equations for the stress in each constituent involve the strains in both constituents. They employed this first-order mixture theory to study time-harmonic wave propagation in both waveguide type laminated and fiber-reinforced composites. Dispersion relations were obtained. A comparison of the approximate and the exact phase velocity data indicates that the theory provides good first mode agreement. Additional accuracy may be obtained by taking more terms in the asymptotic expansions which result in a more complicated higher-order mixture theory.

Hegemier and Bache [H10] further extended the first-order interacting continuum theory for wave propagation in one-dimensional elastic laminated composites to the general two-dimensional elastic case. The phase velocity spectrum of the general two-dimensional theory was investigated for one-dimensional harmonic wave propagation at various propagation angles with respect to the laminates. Comparison with the first three modes of the exact theory gives excellent agreement. It is seen that this theory exhibits somewhat better correlation than the effective stiffness theory, especially at propagation angles close to the normal.

Nayfeh and Gurtman [N4] employed the first-order binary mixture theory to study harmonic shear wave motion in laminated composites. Both transversely (SV) and horizontally (SH) polarized waves were considered. Dispersion relations were derived and compared with the results from the exact and other approximate theories. Comparison indicates that the agreements with the exact theory for both types of shear waves are good, especially for the SH type wave, for quite a wide range of wave numbers, and that the superiority of this theory to effective stiffness is obvious.

Gurtman, Nayfeh and Hegemier [G13] generalized the first-order theory of interacting continuum to study the two-dimensional wave propagation in composite materials. In the generalization, the material effect, the elastic-plastic behavior and two-dimensional lay-ups (laminates) were considered. The dispersion relation of longitudinal harmonic waves in the elastic composite was put in a numerical dispersion code. The predicted response agrees well with the experimental results.

Gurtman, et al. [G14] further applied the theory of interacting continuum to wave propagation in three-dimensionally reinforced composites. The proposed model is an extension of their two-dimensional laminate and

fiber-reinforcement formulations with an inclusion of the effects of thermo-
dynamics, finite deformations and nonlinear constitutive behavior. A DISP
code was developed to study the propagation of harmonic longitudinal waves
in three-dimensional quartz-phenolic composite. Dispersion data obtained
from the DISP code were compared with experimental results. Agreement of
all values within the region of validity (for frequency $\leq$ 1 Mkz) of the
theory is quite good.

Nayfeh [N5] used the theory of continuum based on asymptotic expansions
to investigate time-harmonic waves propagating normal to the layers of multi-
layered periodic media. By retaining all terms in the asymptotic sequences,
he obtained the dispersion relation which yields the exact phase velocity
spectrum. The known relations for the homogeneous and the bilaminated media
are obtainable as special cases from this theory. Also, it is noted that the
dispersion of the composite increases with each additional layer in each
cell of the composite.

Nayfeh [N7] proposed a continuum mixture theory of heat conduction in
laminated waveguides. Theory leads to simple governing equations for the
actual composite which retain the integrity of the diffusion process in each
constituent but allow them to coexist under some defined interactions. The
resulting equations were used to study the harmonic temperature pulse. The
results were found to correlate well with some existing exact solutions.

It was also mentioned before that Ben-Amoz [B22,B23] proposed a con-
tinuum theory for waveguide type propagation of stress waves in fiber-
reinforced composites, using direct asymptotic expansions. Based on asymptotic
expansion technique, he [B24] also developed a continuum theory for wave
propagation in layered composites. Waves propagating normal to the planes
of lamination were considered. A pair of fiber and matrix laminae were

selected as the fundamental unit of the periodic medium. Starting from three-dimensional equations of motion with order of magnitude arguments, a combination of thickness expansion and crystal lattice techniques was used to derive continuum equations (of zero order). The theory was then used to study harmonic wave propagation in the laminated composite. The phase velocity of waves propagating transversely to the laminate was calculated. No numerical example nor dispersion results compared with other theories were reported.

Ben-Amoz [B25] also considered heat conduction in laminated composites by use of the asymptotic expansion technique. Expanding asymptotically the microstructure equations in terms of a small parameter given by the ratio of diffusivities of the two constituents, he again developed a continuum model for heat conduction in which microstructure effects appear as a consequence of the fact that the current state in a laminated medium is history-dependent. Such a history-dependence reflects the effects of microconduction occurring within the microstructure on the micro-time scale.

II.6 Other Methods

Based on perturbation theory and on the statistical consideration assuming elastic parameters of the medium to be subjected to small random fluctuations, Hudson [H20] discussed the scattering of surface waves due to a randomly inhomogeneous medium. He derived expressions for the attenuation of Love and Rayleigh waves when the size of inhomogeneity is small compared with the wavelengths of incident waves. These expressions have the same forms as the known expressions for the viscoelastic attenuation. Comparing the two, he found that the attenuation mechanism by scattering is just like a certain type

of viscoelasticity and that the scattering is more effective at high frequencies and viscoelastic attenuation is more effective at low frequencies.

Based on the assumption that geometric dispersion results mainly from the relatively periodic arrangement of the reinforcement elements in the matrix rather than from the precise shape of each reinforcing element, Drumheller and Sutherland [D10] developed a lattice model which ignores the shape of the reinforcing elements but preserves their periodicity. In the application of this lattice model, the composite was initially treated as a nondispersive homogeneous mixture. The effective or average properties of the mixture were determined by either steady-wave analysis or appropriate experiments. A lattice was then formed by redistributing the mass within the mixture to form a periodic structure of laminates. This mass redistribution was carried out in a manner which yields a lattice with theoretical dispersive charac- teristics that match the measured dispersive characteristic of the composite. Hence, for harmonic wave propagation, this model describes very well the behavior of actual engineering composites.

Nelson and Navi [N11] also used a lattice model to study the harmonic waves of plane strain propagating in composite materials. The medium was considered as a periodic assemblage of identical cells. Using a finite element technique, the dynamic behavior of one reference cell was represented by a number of generalized displacements. Thus, the continuum periodic structure of the medium was idealized by a discrete lattice-type structure and the problem was reduced to an algebraic eigenvalue problem. They deter- mined the eigenvalues for several examples. The frequency spectrum for a fiber-reinforced composite medium was compared with the results of effective modulus and effective stiffness theories. This theory predicts correctly the vanishing of group velocity for half-wavelength equal to the interfiber

distance. The effective stiffness theory yields a cutoff at a smaller wave number while the effective modulus theory fails to predict any dispersion. Hence, the results, when compared with available analytical and numerical results, show that the method gives an accurate model which is able to model periodic structures of wavelength smaller than the lattice constant.

Habip [H1] applied the micromorphic theory of continua to examine the time-harmonic waves propagating in a composite with elastic micro-inclusions. By analogy with the results of Eringen and Suhubi [E2] and of Mindlin [M4], he obtained explicit dispersion relations, governing the lowest and next higher modes of propagation of plane logitudinal waves in an unbounded elastic composite solid, in terms of the relative properties of the constituent materials. The corresponding ratio of group velocity to phase velocity was likewise evaluated. Results valid for a special case were exhibited in graphs.

Ziegler [Z1] considered the mean waves in laminated random composites. He used a simple version of the effective stiffness theory for laminated media in deriving stochastic displacement equation of motion. Application of the perturbation procedure of Keller [K2], modified by introducing the spectral densities of the random coefficients, then led to deterministic equations for mean wave propagation. He gave special attention to uncoupled modes of mean waves propagating perpendicular or parallel to the direction of the layering. He established dispersion relations for these plane harmonic wave trains.

Krumhansl [K9] also discussed the randomness and average or mean wave propagation in inhomogeneous media. He employed the methods used in solid state physics for substantially disordered alloys to study the substantially disordered composite. For such composites, he substituted Fourier-Floquet methods for Fourier plane wave methods. In fact, his approach is a perturba-

tion method making use of Floquet solutions already developed for an average medium. From this theory he predicted: (1) there will be well defined average Floquet waves, but in general they will be damped out in time; (2) the low frequency average waves will be damped the least; (3) the dispersion curves are shifted from those of the periodic medium; (4) moderate homogeneous randomness does not destroy Floquet properties in the average waves; (5) some possibly needed information, such as the fluctuations from the average field and the scattered field and the local information, is lost by the proposed procedure.

McCoy [M2] studied harmonic wave propagation in disordered composites. He obtained a formulation that is to be satisfied by the statistical average field quantities in a statistical sample of heterogeneous, linearly elastic solids. A low-frequency long wavelength theory then was extracted from the general formulation. The predictions of this theory can be given a purely deterministic interpretation. Some special cases of his theory reduces to an effective modulus theory. However, by retaining correction terms, it was shown that elastic wave propagation will always exhibit both dispersion and decay over large enough propagation distances.

Bose and Mal [B41] also employed the statistical approach to discuss the longitudinal shear waves in a fiber-reinforced composite. Waves are harmonic, fibers are randomly distributed in parallel. The composite is statistically uniform. They obtained the phase velocity and damping of the average waves as functions of the statistical and the mechanical parameters. The theory leads to Hashin and Rosen's formula [H6] for the axial shear modulus if the correlations in the positions of fibers are ignored. The correlation terms have a significant effect on the damping property of the composite, especially at high frequencies and concentration. The effect of

the correlation terms is to increase the velocity and decrease the specific damping capacity.

Bose and Mal [B42] again discussed the time-harmonic plane waves in a fiber-reinforced composite by statistical approach. The composite consists of a homogeneous isotropic medium containing long parallel randomly distributed circle fibers of identical properties. They discussed the general case of time-harmonic plane waves propagating perpendicular to the fibers. The main result was to obtain the phase velocity and the damping of the waves of both compressional and shear types into which the average waves are to be separated. In the case of thin fibers (compared to wavelength), the results concerning the transverse bulk modulus and transverse rigidity were compared with those of Hashin and Rosen. Numerical calculations were made for boron filters in an aluminum matrix. It was found again that correlation terms or the correlations in the positions of fibers have a significant effect on the damping property of the composite, especially at high frequencies and concentrations.

As mentioned earlier, Chao and Lee [C3] developed a discrete continuum theory for periodically layered composite materials. They presented the governing field equations incorporating interface continuity conditions in the form of a system of differential-difference equations. They applied this model to discuss the propagation of plane harmonic waves in an unbounded layered medium. Thickness twist vibrations were studied. Numerical results were compared with those of exact solution (harmonic waves). It was seen that the results agree with exact results quite closely. Agreements are better than effective stiffness theory in general as wave number gradually increases.

It was also mentioned before that Ben-Amoz [B30] extended Mindlin's theory of elasticity with microstructure to a heterogeneous medium consisting of inclusions of arbitrary geometry embedded in a matrix material. He applied

this theory to examine harmonic plane wave propagating in such a medium. The dispersion curves for a cubic array of spherical particles obtained by this theory are rather similar to the curves sketched by Mindlin [M4] for the lowest acoustic branches. Certainly, the theory developed by Ben-Amoz is an effective tool in treating wave propagation in media lacking periodicity such as inclusions of arbitrary geometry embedded in a matrix material.

III. TRANSIENT WAVES

In this section, transient analyses of waves propagating in composite materials based on Fourier synthesis are first discussed. Then, in addition, other techniques for obtaining the solutions to transient problems are mentioned.

III.1 Fourier Synthesis

In discussing a pressure pulse propagating in heterogeneous materials, Osten [O2] used a computer code called FURRY. He expressed the applied pressure pulse at $t = 0$ as a Fourier series in time. The amplitude of each component in the Fourier series were then determined numerically. To determine the pulse shape at a later time t, each component of the initial pulse is allowed to propagate the distance ct, where $c = c(x)$ and x is the circular frequency of that Fourier component. The differences in c thus cause the pulse to disperse. Osten also allowed for attenuation of each component in the form $e^{-\gamma t}$ where γ is real and also $\gamma = \gamma(x)$. Osten applied the FURRY code to the sinusoidal dispersion solutions he obtained for a composite with randomly distributed inclusions of spheres and cylinders. He compared FURRY code predictions with the attenuation curves obtained by Bjork [B39], by a series of 2-D finite difference code calculation, and found that FURRY underpredicted the attenuation. He attributed the discrepancies to absence of nonlinear attenuation effects in FURRY and speculated that nonlinearity, acting in conjunction with geometric dispersion, may tend to enhance the dispersive attenuation.

Peck and Gurtman [P3] carried out Fourier analysis of pulse propagating parallel to the interfaces of a laminated composite. They considered

a uniform pressure pulse of step-function in time applied to the boundary of a half-space. The laminates are perpendicular to the half-space boundary. Starting from the equations of elasticity, they applied Fourier integral transforms over time t and the propagation direction x in the analysis. Exact solutions were obtained in the form of an infinite series of integrals, each of which is the contribution to the transient response from a mode of sinusoidal wave.

Peck and Gurtman then evaluated the asymptotic solution in integral form by using the saddle-point technique developed by Folk, et al. [F2]. Of particular importance is a contribution from the first mode, which dominates the long-time solution. This contribution is known as the head-of-the-pulse approximation. For the strain in the direction of propagation, the head-of-the-pulse solution takes the form

$$\varepsilon_x(x,y,t) \sim \varepsilon_0 H(\xi) \tag{46}$$

where

$$H(\xi) = \frac{1}{3} - \int_0^{-\xi} A_i(\eta)d\eta , \tag{47}$$

$$\xi = (t - x/c_0)/\tau , \qquad \tau = (3c_0^{-2}\alpha x)^{1/3}$$

in which c_0 is the speed of wave propagation, α is the crucial dispersivity parameter, ε_0 is the static strain under the same load and A_i is the well-known Airy function. The above expressions can be interpreted as follows: the wave is roughly a step pulse arriving at $\xi = 0$, i.e., at $t = x/c_0$, so that the main disturbance is propagating at speed c_0. The quantity τ has the dimension of time, and $t - x/c_0$ is time after arrival

of the wave. Thus ξ is a nondimensional time of arrival. The quantity τ may be regarded as a characteristic dispersion time of the pulse.

Equation (46) states that the strain is uniform in the direction of propagation. The stress in the x-direction is quite non-uniform because the moduli of the layers are different. In terms of the average stress over the cross section, $\bar{\sigma}(x,t)$, one has

$$\bar{\sigma}(x,t) \sim \sigma_o H(\xi) \tag{48}$$

where σ_o is the applied stress.

Further applications of the head-of-the-pulse approximation will be mentioned later when it is employed in other problems.

Voelker and Achenbach [V1] carried out a similar analysis of stress waves in a laminated medium generated by transverse forces. The laminated medium composed of alternating layers of two homogeneous isotropic elastic solids is suddenly subjected to a spatially uniform distribution of transverse forces, which are applied in a plane normal to the layering. The resulting two-dimensional transient-wave propagation problem is analyzed by means of modal analysis. The normal and shear stresses at the interfaces are expressed as infinite integrals that are integrated for not too large values of time. For large values of time, the integrals are estimated by the method of stationary phase. The predominant contribution to the interface shear stress comes from the head-of-the-pulse approximation. The normal stress at the interface, which is

composed of several contributions, is oscillatory (and dies out at $t^{1/3}$),
and thus the interface bonds may be subjected to tensile stress. The solu-
tion obtained should be useful in determining the validity of the continuum
theories.

Balanis [B3] considered the transient one-dimensional wave propagation
in a semi-infinite periodic composite medium, which consists of periodic
array of two elastic layers perfectly bonded together. Waves are generated
by a surface velocity input and travel perpendicular to the layers. He
applied the Fourier integral transform over time t in the analysis.
Fourier transformed interface velocities and stresses are analyzed. An
approximation for materials with similar impedences is then presented. He
found that for situations where the times taken by the longitudinal wave
to traverse the layer thicknesses are in an integer ratio, periodicity
results in the frequency domain. He treated such a case and obtained
the interface stressed due to a square wave input. The results of
this example are quite interesting. However, the approximation scheme
seems to suffer from lack of generality.

III.2 Floquet or Bloch Theories

Krumhansl [K8] analyzed the propagation of transient waves in
periodic composites by means of Floquet or Bloch theory. He used the quasi-
periodic Floquet waves to form a basis for the analysis of transient problems
and expressed the solutions of the problems in the form of Fourier-Floquet
series. A brief summary of this theory for one-dimensional systems is as
follows:

Consider the Floquet wave solutions for displacement

$$u_\nu(x;q) = v_\nu(x;q) \exp(iqx) \tag{49}$$

with time variation prescribed by the additional factor $\exp(\pm i\omega t)$, where

$$\omega = \omega_\nu(q) , \tag{50}$$

q is the wave number, and ν represents the νth mode.
The function $v_\nu(x;q)$ is periodic with the period of the composite configuration. The solutions are defined in ν denumerable regions of ω, the pass band, and for $-\pi/a \le q \le \pi/a$ where a is the width of a unit cell. Multiplying constant factors can be selected for the u functions to generate an orthonormal set satisfying the relation:

$$\int_{-\infty}^{\infty} \rho(x)u_\nu^*(x;q)u_{\nu'},(x,q)dx = \delta_{\nu\nu'},\delta(q-q') \tag{51}$$

The band orthogonality is given by the Kronecker delta and the q space orthogonality by the Dirac δ-function. Using this relation, solution for propagation of waves without applied forces can be represented in the form:

$$u(x;t) = \sum_\nu \int_{-\pi/a}^{\pi/a} c_\nu(+,q)e^{i\omega_\nu t} + c_\nu(-,q)e^{-i\omega_\nu t} \, u_\nu(x;q)dq \tag{52}$$

with the coefficients $c_\nu(\pm,q)$ determined by application of orthogonality:

$$c_\nu(+,q)e^{i\omega_\nu t} + c_\nu(-,q)e^{-i\omega t} = \int_{-\infty}^{\infty} \rho(x)u_\nu^*(x;q)u(x;t)dx,$$

$$\tag{53}$$

$$i\omega_\nu\left[c_\nu(+,q)e^{i\omega_\nu t} - c_\nu(-,q)e^{-i\omega_\nu t}\right] = \int_{-\infty}^{\infty} \rho(x)u_\nu^*(x;q)\dot{u}(x;t)dx$$

Equation (52) thus provides the solution for an initial value problem when the initial displacement and velocity are prescribed. For initial disturbance in a localized region, this method enables one to manipulate the solution

into the form of a stationary phase integral for which a head-of-the-wave analysis can be carried out. It is also noted that by separation of solutions into components even and odd in x, half-space problems can be treated.

Krumhansl and Lee [K10] further employed the Fourier-Floquet methods to transient elastic waves in periodic composites. They found that the head-of-the-wave solutions in the far field are dominated by the maximum group velocity contributions from each frequency band. For an infinite periodically layered medium which is initially at rest and subjected to a momentum impulse at x = 0 (it would be convenient to arrange x = 0 at symmetry plane of the periodic composite), the velocity response at far field is given in closed form by a set of Airy functions. The appearance of Airy functions in the far field response is just like what Peck and Gurtman obtained in the head-of-the-pulse approximation for an analogous problem. In addition, Krumhansl and Lee found the asymptotic behavior of a pulse initially uniform over a plane with the following features: (a) the asymptotic peak amplitude decreases as $x^{-1/3}$, (b) the pulse spreads out and becomes less steeper with rise time proportional to $x^{1/3}$, (c) peak stress also falls off at large x, though it is complicated to calculate. They also discussed the motion of a periodic layered composite under applied forces. The discussion includes both a formal representation of the Green's function and the treatment of short pulse pressure loading on a surface.

A Fourier-Bloch technique for superposing the finite-element sinusoidal Bloch wave modes to generate transient solutions to 2-D boundary value problems was also presented by Peck, et al. [P4]. For a step-pressure load on the surface of the composite, the responses are calculated at small and

large propagation distances. Five modes give good convergence at small propagation distance, about 1/2 spacing of the longitudinal layers, although even more modes would be desirable. However, at the larger propagation distance (about twice the spacing of the longitudinal layers) one mode only provides a good approximation to the response. This dependence of number of modes required on the propagation distance provides some insight into the regions of validity of one-mode or two-mode approximations that are commonly obtained with the continuum theories. The transient solutions also showed that the layering in the propagation direction changes the micromechanical shear stresses at the edge of the longitudinal layer by a factor of two, even though the layering has little effect on the overall dispersion.

Kohn [K6] considered the propagation of low-frequency elastic disturbances in an infinite one-dimensional composite which is of periodically varying density $\rho(x)$ and stiffness $\eta(x)$, with spatial periodicity a. By using Floquet theory and low frequency expansions, he found that in the limit of low frequencies the displacement $u(x,t)$ in the composite can be written in the form of a differential operator acting on a slowly varying envelope function $U(x,t)$ as

$$u(x,t) = [1 + v_1(x)\partial/\partial_x + \ldots]U(x,t) \tag{54}$$

$U(x,t)$ itself describes the overall long wavelength displacement field. It satisfies a wave equation with constant (independent of x) coefficients obtainable from the dispersion relation $\omega = \omega(k)$ of the lowest band of eigenmodes, i.e.

$$(\partial^2/\partial t^2 - \bar{c}^2 \, \partial^2/\partial x^2 - \beta \, \partial^4/\partial x^4 + \ldots)U(x,t) = 0 \tag{55}$$

where $\bar{c}$ is the long wavelength sound velocity of the composite and β is the 4th-order expansion coefficient of the square of the frequency of the lowest band of eigenmodes

$$\omega^2(k) = \bar{c}^2 k^2 - \beta k^4 + \dots \tag{56}$$

$v_1(x)$ in (54) is given by

$$v_1(x) = \int_{a/2}^{x} \left(\frac{\eta(x')^{-1}}{\overline{\eta^{-1}}} - 1 \right) dx' \tag{57}$$

where

$$\overline{\eta^{-1}} = \frac{1}{a} \int_0^a \eta^{-1}(x) dx \tag{58}$$

Information about the local strain on the microscale of the composite laminae is contained in the function $v_1(x)$. Kohn then applied this approach to study the pulse propagation in a half-space. Appropriate Green's functions are constructed in terms of Airy functions. With these Green's functions, the solution of the initial value problem for a composite half-space with free boundary is explicitly obtained. It is noted that this long wavelength theory correctly describes the head of a resulting pulse, even if the prescribed initial pulses have step-function character.

Kohn [K7] extended the previous work to three-dimensional composite material. In this case the displacement solutions to equations of motion of periodic elastic medium are expressed as a vector in the form of a differential operator acting on a vector function which describes the mean displacement of each cell. Again, local strain information can be obtained from the solution. One of the applications of this method is the structure of the head-of-a-pulse propagating in an arbitrary direction. However, application of

solutions to fiber-reinforced materials is conceptual rather than explicit. His approach may be regarded as a method of microscopic field, although relation of his work to other dynamic theory was not given.

III.3 Effective Modulus Theories

Aboudi and Weitsman [A2] discussed the two-dimensional plane problem of an impact on a half-space reinforced by parallel elastic fibers with a pulse of finite duration. They assumed that the rigidity of the fibers is greater than that of the matrix. Based on effective modulus theory, as described previously, they expressed the constitutive equations in terms of the reinforcement ratio and angles of inclination of the fibers. Solutions were obtained by a finite difference scheme. Numerical results were presented for the case of normal impact on a half-space. Comparison with the isotropic case showed good accuracy of the solutions (with maximum error less than 1/2 percent). The investigation indicates that the vertical displacement amplitudes decrease monotonically with fiber orientation. However, the horizontal displacement amplitudes are shown to vary with position with maximum and minimum displacements less predictable as a function of fiber orientation.

Aboudi and Weitsman [A5] extended their approach to study the problem of an impacted fiber-reinforced viscoelastic half-space. The matrix, they considered, is an arbitrary linear viscoelastic homogeneous and isotropic half-space. The embedded oblique fibers are linear elastic, arranged in parallel order and randomly dispersed. They adopted the effective modulus theory to study the dynamic response of the composite half-space to a time dependent surface load. Again, a numerical procedure of finite-difference type was applied to the set of equations resulting from the theory. Results were worked out for an elastic-elastic half-space and a viscoelastic epoxy-elastic glass fiber half-space. Stability, convergence and accuracy of the numerical procedure

were considered. Numerical results for surface displacements and displacements at some distances beneath the surface were obtained and shown graphically. Since effective modulus theory only gives weighted displacements and stresses, it is highly possible that the effective stiffness theory or interacting continuum theory, which accounts for microstructure information, might give more reliable numerical results.

Eason [E1] analyzed the propagation of waves resulting from the sudden application of a force to the surface of a cylindrical cavity in an infinite fiber-reinforced solid. The fibers were assumed circumferential to the cavity and uniformly distributed so that the material might be regarded as a transverse isotropic solid. He used an effective modulus model to represent this solid. The stress distribution within the solid then was determined numerically. He found that the maximum of the tangential stress, σ_θ, at the cavity increases from a value which is slightly greater than the suddenly applied force Q for the isotropic solid to a value of several times of Q for a highly anisotropic solid. He also found that the time at which this maximum is achieved decreases from the time for isotropic solid to a fraction of that time for highly anisotropic solid.

Sun, Feng and Koh [S28,S31] investigated the shear wave propagation in a nonlinear elastic fiber-reinforced composite. The composite was modeled by a medium consisting of thin nonlinear (isotropic) matrix layers alternating with the effective linearly elastic fibrous layers. Using the effective modulus theory, they derived a set of three-dimensional nonlinear constitutive equations. The model was employed to investigate the wave front propagation, the stability of the shock front, growth and decay of the shock wave and the distortion of initially sinusoidal wave. Due to the

nonlinearity, a sinusoidal shear wave is distorted. At some distance a discontinuity may occur at a critical time. Shock front may form at such discontinuity from an initially harmonic wave. It is noted that shocks can be formed for composite with hardening matrix but cannot be formed for that with softening matrix.

III.4 Effective Stiffness Theories

Sve [S40] discussed the propagation of a shear pulse parallel to the interfaces of a periodically laminated medium. The propagation of wave was caused by impulsive shear loads perpendicular to the laminate. He employed the effective stiffness theory to model the composite and applied the method of the head-of-the-pulse to obtain an approximate solution. It turns out that the solution, say for particle velocity normal to the laminates, is the head-of-the-pulse function for impulsive stress normal to the laminates [P3] reversed, with oscillations of increasing amplitude preceding the main pulse followed by a monotonic rise to a nondimensional amplitude of unity.

Sve and Whittier [S41] also investigated the one-dimensional pulse propagation in an oblique laminated half-space. Using the effective stiffness theory and the head-of-the-pulse approximation again, they studied the impulsive waves in an oblique laminate caused by a step-pressure loading on the boundary. It is interesting to note that the response of an oblique laminate is both anisotropic and geometrically dispersive. The anisotropy leads to a two-step response to a step loading, while the dispersivity smooths off the two steps and causes oscillations preceding or following the steps. They found that the character of the response changes as the angle of laminate, α, is varied. When $\alpha = 0$, a single pulse propagates

through the composite; as α is increased, the pulse splits into two; when $\alpha = 90°$, the response is once again a single pulse.

Sve [S45] also applied the effective stiffness theory to the response problem of a periodically laminated elastic half-space subject to rapid internal heating. Propagation direction parallel to the planes of the laminates was considered. The analysis indicates that dispersion assumes an important role in thermally induced stress waves. The pulse in the far field may bear little resemblance to the pulse that originated at the front surface.

Sve and Herrmann [S4] also applied the effective stiffness theory to study the dynamic response of a periodically laminated half-plane subjected to a moving load. The laminations are parallel to the surface of the half-plane and the traveling load is a step load of magnitude P inclined at an angle ϕ to the surface. The velocity of the load is constant and supersonic. They obtained the formal solution with the aid of Laplace transforms and constructed a far-field solution with the head-of-the-pulse procedure. They calculated the gross normal strain at the rear face for: (1) moving normal load ($\phi = \pi/2$), (2) moving shear load ($\phi = 0$). The results indicate that the primary effect of the laminations is to create a strain that has a finite rise time and is oscillatory about the value obtained on the basis of the effective modulus theory which predicts a two-step response for $\phi = 0$ and a rectangular response for $\phi = \pi/2$. Also, for normal loading tensile strain will occur shortly after the arrival of the shear wave.

Whitney and Sun [W14] presented a refined effective stiffness theory for extensional motion of laminated composites. In this theory, terms of second order in z which is the coordinate in the direction of the thickness of laminate are included for the longitudinal displacements while only the first order terms in z are included for the flexural displacements. The analysis is appropriate for oblique impact loading. In comparison with other theories, dispersion curves appear to verify this refined theory. With some modification, the theory is applicable to fiber-reinforced composite materials.

Bedford and Drumheller [B16] further discussed the higher-order effective stiffness theory. A set of displacement and stress boundary conditions at an external surface compatible with the higher-order theory were presented. The method developed can be used to obtain boundary conditions for theories of arbitrary order. Also, it makes explicit the relationship between the microdisplacement and microstress distributions and the actual displacement and stress distributions on the surface of the composite.

Bedford and Drumheller [B1] applied a second-order effective stiffness theory for the dynamic behavior of elastic laminates to the problem of a laminated half-space, with interfaces normal to the boundary, subjected to harmonically time varying displacement and stress distribution at the boundary. The finite number of modes of the microstructure theory were found to be sufficient to model a uniform normal displacement boundary condition but not a uniform normal stress boundary condition. The solutions yield the constituent displacement and stress distributions both near the boundary and in the far field and permit an assessment of the usefulness of the microstructure theory for such boundary value problems.

The first nonlinear theory of the effective stiffness type was obtained by Rausch [R3] who analyzed the transient wave propagation parallel to the interfaces of a laminate. In deriving the effective stiffness equations, he expressed Hamilton's principle in terms of a potential energy, which

results in nonlinear constitutive properties, and a kinetic energy containing large strains and displacements to produce a kinematically nonlinear theory. He then sought the solution to the effective stiffness equations by a coordinate-perturbation technique based on asymptotic expansions [R2] in stress amplitude. The equations for the first-order perturbations are linear differential equations which can be solved by integral transform techniques. He approximated the transformed solutions for large times by use of the saddle-point technique. Thus, he obtained a nonlinear head-of-the-pulse approximation. The first-order theory presented shows how non-linear effects cause the geometrically dispersed wave to form a shock but does not give the second-order effect in which dispersion limits the steepness of the wave. Consequently, steady waves of the type found experimentally cannot form. Thus a further approximation will require the development of a second-order theory. However, the results of the first-order theory provide an insight into the nature of the interaction between geometric dispersion and nonlinear constitutive effects. It appears both nonlinearity and geometric dispersivity have the effects to cause early attenuation.

III.5   Interacting-Continuum Theories and Mixture Theories

Based on the interacting continuum concept in which every constituent has its own motion but is allowed to interact with others, Bedford [B12] developed a basic nonlinear theory for composites and explored the implications of this theory with regard to shock wave propagation. He found that each constituent can support separate shock waves so that multiple shock waves can occur in such a theory. Such multiple shock waves may be more an artifact of the theory than an essential part of the modeling, and probably are similar to the characteristic wave speeds in the approximate theories

of rods and beams, which are not necessarily closely associated with the calculation of the main response. In this theory, the appropriate boundary conditions and Rankine-Hugoniot equations for a multi-continuum model of composite material were formulated.

Bedford [B15] extended the multi-continuum theory to formulate the jump conditions for mass, momentum and energy across a singular wave surface propagating in a multi-constituent bonded-elastic composite. He found that the shock surfaces, linked by the reference configuration, exist for each constituent. The results are relevant to the study of acceleration waves and shock waves and may be adopted to give boundary conditions. The method has been applied successfully to single continua and can be applied to a broad class of continuum theories.

Hegemier and Nayfeh [H7] developed a continuum theory based on an asymptotic expansion for wave propagation in laminated composites in which dominant signal wavelengths were assumed to be large compared to typical composite micro-dimensions. The case of propagation normal to the laminates was considered. They obtained a hierarchy of models defined by the order of truncation of the asymptotic sequence. In principle, retention of all terms in the asymptotic sequence yields the exact theory. They then cast the lower-order dispersive theory in a standard mixture form. They obtained the transient pulse responses at the centroids of the 15th layer and 12th layer respectively due to a step-stress input on the boundary. Comparison of the result with exact analysis obtained from a one-dimensional numerical code shows quite good agreement. It is noted that the one-third points (1/3 first peak) on the transient waves travel at the mixture velocity. A similar result was found in the head-of-the-pulse analysis of Peck and Gurtman [P3].

Hegemier, Gurtman and Nayfeh [H8] extended the interacting continuum theory by a rational construction technique for the constitutive equations and interacting term. They considered the case of waveguide-type propagation in laminated and fiber-reinforced composite. Utilizing mixed spatial and asymptotic expansions, they found that their asymptotic approach leads to a first-order expression for the momentum interaction term identical in form with Bedford's [B11], but the constitutive equations for each constituent involve the strains in both constituents. Hence, the resulting first-order continuum mixture theory contains microstructure. The utility of this theory demonstrated for both laminated and fibrous composites by correlating theoretical and experimental [W15] transient pulse data on boron-carbon phenolic and Thornel-carbon phenolic laminated, and unidirectional fibrous quartz-phenolic. In each case, the input pulse is a step function in velocity applied to both constituents at the boundary. The response at the rear surface is a steady rise of amplitude followed by oscillations. In all cases, the agreements between theoretical and experimental results are quite good.

Hegemier and Bache [H9] again discussed the transient pulse propagation, parallel to laminates in elastic-laminated composites. Based upon asymptotic expansions they developed a modified first-order mixture theory. For a step velocity input on the boundary, the transient data obtained from the simplified theory exhibits good correlation with experimental results. Because of its simplicity and satisfactory prediction, this simplified theory, they hoped, will lead to maximum utility.

Bache and Hegemier [B1] further discussed the transient wave propagation in viscoelastic-laminated composites. With minor modifications of the techniques used for elastic cases, they constructed a general theory for wave propagation in viscoelastic-laminated composites. The procedures are

quite straightforward. They first applied the Laplace transformation to the equations of motion and viscoelastic-constitutive relations, then they deduced the continuum theory for the problem under consideration in the Laplace transform plane from the well-known correspondence principle. Finally, they took Laplace inversions and obtained the theory in the physical space. In view of its simplicity and potential utility, they derived the corresponding viscoelastic modified first-order theory, and applied this simplified theory to investigate the transient pulse normal to the laminates due to a step-velocity input on the boundary of a half-space formed by the bilaminates. The composite was laminated by a quartz fiber bundle in a phenolic matrix (constituent 1) and Ironsides FF-17 phenolic resin (constituent 2). The quartz fiber material was modeled as elastic for propagation normal to the fibers while the standard linear solid model of viscoelasticity was used for the phenolic resin material. They compared the stress-time history at the midplane of the 13th layer to that for a corresponding linear elastic composite. It is seen that the general shape of the former is more or less similar to that of the latter but with the amplitude of the peak stress and the oscillations following the peak damped out somewhat.

Nayfeh, Gurtman and Hegemier [N3] used the theory of interacting continuum to study elastic wave propagation normal to the fibers of unidirectionally-reinforced composites. The composite consists of a periodic array of square fibers which is perfectly bonded to the matrix material. They derived a "first-order" type dispersive model for this complex geometry. The basic construction procedure is as follows: (1) they modeled the fiber-matrix regions as a mixture using the same procedure for layered materials; (2) they modeled the matrix

material regions as homogeneous laminates under unidirectional motion in the direction normal to the layers; (3) they then followed the same procedure as that for obtaining wave propagation normal to layered composites to conctruct a dispersive continuum theory for the fiber reinforced composites. The simplified first order continuum model so constructed is seen to be similar in form to the theory obtained by them for propagation normal to the layers of a laminated composite. The only difference, in fact, is the complexity of the constants in the equations. Utilizing this theory, they obtained the solution for a step-stress input at the boundary of a semi-infinite medium of a unidirectional fiber-reinforced quartz phenolic composite. The step-function pressure was applied to the front surface of the material which was a fiber-matrix laminae. The calculated rear surface velocity, the velocity corresponds to that of a matrix laminae, vs. time curve is again a steady rise of magnitude followed by oscillations. Comparison of experimental data reported by the Aerospace Corporation [H11] and theoretical results showed that agreement is fair. If the geometrical approximation is improved and the visco-elastic nature of the phenolic is accounted for, the agreement should be much better.

Gurtman, Nayfeh and Hegemier [G13] generalized the theory of interacting continuum (TINC) to study the two-dimensional wave propagation in structural composite materials. In their generalization, a caloric equation of state was assumed for mean stress vs. density, and an elastic, perfectly-plastic law of von Mises was assumed for the deviation stress tensor. In addition, two-dimensional lay-ups (laminates) were considered. They then developed a hierarchy of two-dimensional models of interacting continuum defined by the order of truncation of the asymptotic expansion. Again, they focused their attention on the simplest "first-order" theory hoping that its relative simplicity will lead to maximum utility while preserving the desired

micromechanical information required by the composite designers and the
analysts. Transient response to a step-function in velocity at the boundary
of either the elastic medium or the elastic, perfectly-plastic medium
correlates well with the corresponding experimental results.

Nayfeh and Gurtman [N4] extended the interacting continuum theory to
study the transient shear wave motions in laminated waveguides. Both trans-
versely (SV) and horizontally (SH) polarized waves were considered. Transient
solutions for both types of waves were investigated via the head-of-the-pulse
approximations. The transient response of the SH waves to a step-boundary
velocity w_o is a pulse beginning with a steady rise, followed by oscilla-
tions about the input boundary velocity, while the response of the SV waves
to a step-boundary velocity v_o is a pulse starting with oscillations about
zero followed by a smooth rise to its final value (the boundary input). It
is noted that: (1) the transient solution of SV wave is identical to that
obtained by Sve [S40] based upon the effective stiffness theory, and (2) the
transient solution of SH wave closely parallels that of the longitudinal
wave solution of Peck and Gurtman [P3] obtained by the exact theory and that
of Hegemier, et al. [H8] obtained by the continuum mixture theory.

Gurtman, Nayfeh and Hegemier, et al. [G14] applied the theory of inter-
acting continuum to study wave propagation in three-dimensionally reinforced
composites. The model developed is an extension of previously developed two-
dimensional laminate and fiber TINC formulations and explicitly considers
the effects of thermodynamics, finite deformations and nonlinear constitutive
behavior. The three-dimensional material is made up of "radial" and "lateral"
fiber bundles in an orthogonal array, with pockets between them filled with
pure resin. They treated the fiber bundles as isotropic, but using their
material properties in the direction of propagation. Thus, material properties

along the fiber are assigned to the radials, and those transverse to the fibers are used for the laterals. Basically, they proposed modeling the unit cell of three-dimensional quartz-phenolic as a combination of cylindrical and laminate waveguides. The initially square cross section of the radial fiber is transformed into a cylinder of equivalent volume fraction surrounded by an annular sheath whose material properties are dynamical and dependent upon those of the alternating fiber and resin pocket. For the cylindrical waveguide, the individual constituents are modeled as elastic-perfectly-plastic material, with a Mie-Grüneisen caloric equation of state relating mean pressure, density and internal energy, and with a von Mises' yield criteria and associated flow rule governing the stress deviators. Thus, they developed a thermodynamic theory of interacting continuum, of waveguide type, for finite amplitude elastic-plastic wave propagation in fiber-reinforced composites. They then smoothed the surrounding structural elements of the actual material made up of lateral fibers and resin pockets via the TINC laminate analysis into a hollow cylinder sheath. The complete thermoelastic-plastic, three-dimensional TINC model then is obtained by combining the solid waveguide fiber with the dispersive hollow cyclinder sheath. For elastic pulse propagation in three-dimensional quartz-phenolic, they used the finite difference TINC code to compute the average velocity at the rear of a specimen 0.635 cm thick. A step-boundary velocity is applied to both fiber and sheath. They found that the velocity-time response is again a steady rise followed by oscillations about the input. The TINC code result was compared with experimental data obtained at AFWL (Air Forces Weapons Laboratory), and with the head-of-the-pulse approximation. The agreement is excellent. The head-of-the-pulse solution, while effectively duplicating the peak velocity and rise time of the wave, tends to diverge from the

TINC results following the passage of the front. They also used the TINC code to calculate the stress histories at the rears of three three-dimensional quartz-phenolic specimens of different thickness which were tested at the AFWL using a light gas gun. Comparison of code predictions and experimental results indicated that there appears to be a tendency on the part of the code to slightly overpredict the rise times, but agreement of peak pressure is excellent.

For nonlinear pulse propagation in three-dimensional quartz-phenolic they used the corresponding TINC code to calculate the stress histories at the rears of several 3DQP specimens which were impact-tested at AFWL using a high pressure gas gun. Their numerical results and experimental data indicated again that the TINC code tends to slightly overpredict rise times, but accurately calculate the peak pressures, and significantly diverge from the experimentally determined curves after the peak.

The theory so developed has been shown to be capable of predicting the nonlinear response of 3DQP to both mechanical and thermodynamic loading, and capable of yielding information concerning the behavior of the three-dimensional material's constituents. It should be a useful tool in both design and analysis of three-dimensional composites.

Hegemier [H12] discussed the finite-amplitude elastic-plastic wave propagation in laminated composites. He developed a binary mixture theory for waveguide type-propagation parallel to the layers of a two-constituent laminated composite with periodic microstructure. The model incorporates the effects of thermodynamics, finite deformations, and nonlinear elastic-plastic constituents. For a step-boundary velocity input, transient displacement, stress and internal energy distributions within the micro-components are produced to a certain degree of accuracy.

Hegemier and Gurtman [H13] also investigated the finite-amplitude elastic-plastic wave propagation in fiber-reinforced composites. Accordingly, they developed an approximate nonlinear theory to describe waveguide-type propagation in unidirectional fibrous composites. The model, an extension of the previously developed laminate formulation, considers the effects of thermodynamics, finite deformations, and nonlinear elastic-plastic constitutive behavior. A one-dimensional binary mixture theory is thus followed. Transient wave-propagation solutions due to a step-boundary velocity input were obtained numerically and compared with another solution from a well-known two-dimensional finite difference code. The agreement is judged to be excellent. However, the authors concluded that the TINC calculation is much more time-saving than that of the finite difference code.

Aboudi [A6] extended his previous work [A3] and formulated a mixture theory for a thermoelastic laminated medium composed of two constituents in alternating layers. In this theory, each constituent has its own motion and temperature, but is allowed to interact mechanically and thermally with the others. The resulting system of coupled equations of motion and heat conduction was then used to study the response of a laminated plate subjected to mechanical and thermal impulsive loadings. Comparison between the results of this theory and those based on the thermoelastic effective modulus theory which he formulated exhibits the pronounced effect of microstructure and the effect of the reinforcement volume on the resulting field in the individual constituents. The effect of the reinforcement is to decrease the stress in the matrix and to increase it in the reinforcement.

Aboudi and Benveniste [A7] proposed a superimposed mixture theory for wave propagation in a biaxially fiber-reinforced composite. The medium studied has alternating layers in which the angle of the biaxial fibers

alternates from layer to layer. Each constituent has its own motion but interacts with others. The layering microstructure is taken into account by the interface shear stresses. The composite is then in the form of a laminated medium composed of uniaxial fiber-reinforced material in alternating layers. They modeled each layer by Bedford's mixture theory and modeled the composite by Hegemier's binary mixture theory. In other words, the model is a superposition of the two known theories. They calculated the response of a material consisting of carbon phenolic reinforced by boron fibers for pulse propagation in the direction parallel to the layers. In contrast with the uniaxial ones, numerical results showed that the stresses in the fibers in the case of biaxial reinforcement attain higher values as compared with the uniaxial one. The normal stresses in the matrix are almost identical for both types of reinforcements. It would be of interest to compare the predicted results with experimental work and thus evaluate the validity of the proposed model.

Nayfeh [N8] treated the transient pulse propagation in porous composites. He employed the interacting continuum theory to analyze the influence of inclusions and porosity on the elastic response of both homogeneous and laminated composite media. The general model analyzed by him consists of periodic array of two perfectly-bonded laminates; one of which consists of an elastic homogeneous material while the other is made up of periodic array of cylindrical elastic inclusions that are distributed in another elastic matrix material. He deduced several specific models as special cases. In all cases porosity is simulated in the limit as the properties of the inclusions identically vanish. He demonstrated that porosity plays a major role in the geometric dispersion of such media. In particular it increases the pulse arrival and rise times (spreading) of a transient pulse. For the

special case of elastic inclusions in a homogeneous matrix media, the results correlate very well with existing experimental data and other approximate analyses.

Nayfeh [N7] extended the interacting continuum (or continuum mixture) theory to discuss the heat conduction in laminated waveguides. The resulting theory leads to simple governing equations for the actual composite which retain the integrity of the diffusion process in each constituent but allow them to coexist under some interactions. He then utilized the resulting equations to study the response due to transient loadings. Solutions were derived by means of Laplace transform techniques. Analytical inversion of transforms was carried out only for the limiting cases of "weak" and "strong" thermal coupling. The limit of strong interaction leads to the coalescence of both temperatures; in this limiting case the composite behaves as a single but high-order continuum according to his investigation. For the general coupling cases, his results were obtained by a direct numerical inversion of the transforms. Since his results for harmonic loadings correlate well with some existing exact solutions, and those for transient loadings are reasonable from physical considerations, the theory should be useful in studying any transient temperature pulses in laminated composites.

Gurtman and Hegemier [G15] modified a previously developed binary mixture theory to account for debonding for waveguide propagation in laminated composites with periodic microstructure. In modification, they relaxed the interface shear stress boundary condition and proposed an interface model of the Coulomb frictional-type. They then applied the resulting theory to analyze the response of a waveguide-type laminate to impact loading at the boundary, and compared the results with experimental results. A fairly good agreement was observed.

Benveniste and Aboudi [B32] also extended the mixture theory to account for debonding due to impulsive loading for wave propagation in a laminated medium. Wave propagation is in the direction of the layering of a bi-laminated medium with the presence of imperfect bonding at the interfaces. The debonding is modeled by a flexible bond, i.e., an inertia-less thin elastic film, which was originally proposed by Jones and Whittier [J2]. Therefore, the debonding mechanism is represented by a model which allows imperfect bonding both in the normal and tangent directions. From this, they formulated a modified mixture theory and applied the theory to transient wave propagation in the waveguide-type laminated composite. It was found that the debonding in the tangent direction is significant in modifying the shape and amplitude of the propagating pulse.

As pointed out previously, Ben-Amoz developed a continuum theory of composite materials in a series of papers [B22,23,28,29] using direct asymptotic expansions. The developments of the theory indicate that the dynamic behavior of the model is "nonlocal" in time as a result of the history-dependence of the current state. From this theory he deduced a zero-order model for practical applications. He analyzed the behavior of this zero-order system and found that during an early phase the motion is confined to a boundary layer and consists of highly damped waves. During a later phase the behavior approaches that of a macroscopically homogeneous medium. The behavior during both phases is described by two distinct systems of differential equations. He then applied this theory to determine the early phase behavior of a laminated half-space subject to a step normal load for (1) propagation parallel to the laminates [B28], and (2) propagation normal to the laminates [B29]. The results have no appreciated differences from the head of the pulse solutions predicted by other theories.

Ben-Amoz [B27] further considered elastic-plastic waves in laminated composites. Based on the assumption of perfectly-elastic fiber layers and elastic-plastic matrix layers, a continuum model was deduced for wave motion either parallel to or normal to the direction of layers. The derivation was again based on asymptotic expansions in terms of a small parameter so that a continuum theory was obtained in which microstructure effects appeared. Again he found that during the early phase the motion is confined to a boundary layer and consists of highly damped waves whereas the behavior during the later phase is predominantly that of a macroscopically homogeneous medium. Considering that pulse attenuation occurs mainly during the early phase, he concluded that the early phase motion constitutes the critical phase of motion as in the elastic wave case. The model should be adequate for composites that are strongly inhomogeneous.

III.6 Other Methods and Problems Related to Composites

Stern, Bedford and Yew [S15], and Bedford and Stern [B10] analyzed waveguide propagation in layered and unidirectional fiber-reinforced composites consisted of elastic and viscoelastic materials. They assumed a simplified displacement field consisting of motion only in the propagation direction. The displacements are constant over the elastic reinforcing layer, but vary over the cross section of the matrix. The resulting equations governing such motion were solved for sinusoidal waves and yielded very good comparisons with exact solutions in the case of layered composite. The same form of solution was applied to the fiber-reinforced material with only a geometric correction for the second-order term in the low frequency expansion of the c-ω curve, and yielded good comparison with the exact solution in contrast with the effective stiffness theories where the agreement with the exact theory is best for large ratios of filament to matrix stiffness (see, for example, [S18]).

Payton [P2] investigated the dynamic bond stress in a composite structure subjected to a sudden pressure rise. The composite structure consists of two elastic semi-infinite rods bonded together along their generators. By introducing an interaction term into the one-dimensional wave equations, he obtained the dynamic bond shearing stress for the case of a pressure step suddenly applied over the end of the rod. The bond stress is composed of a static and a dynamic part and the peak bond stress that occurred at the loaded end is dominated by the static part. Results obtained may provide information to designers in minimizing the possibility of bond failure from dynamic loading.

Achenbach, Hemann and Ziegler [A14] studied the tensile failure of interface bonds in a composite body subjected to compressive loads. Carrying out the one-dimensional analysis of wave propagation normal to the interfaces of a laminate, they found that a compressive load can lead to tensile stress when a wave is transmitted across an interface to a lower impendance material, and that fracture conditions also depend on the duration of the load and the thickness of the layers. When a load of duration less than the transit time of either layer is applied to the lower impedance layer, fracture caused by the tensile wave reflected from the second interface will occur at the first interface. This predication was verified by experiments.

Ting and Lee [T8] applied the geometrical acoustics approach to determine the response of embedded circular and spherical inclusions to a plane dilatation wave in the surrounding medium. One of the most interesting results obtained by them is the existence of caustics, which is an envelope of rays, within the embedded medium. Since each ray carries a finite signal, while there are infinite rays meeting at a caustic, the response is infinite at the caustic. This suggests that nonlinear response, possibly including fracture, will occur.

The geometrical acoustics approach similar to that used by Ting and Lee was applied by Achenbach, Hemann and Ziegler [A17] to investigate the separation at the interface of a circular inclusion and the surrounding medium under an incident

compressive wave. They discussed the nature of the singularity associated with the caustic. The singularity is of a square root dependence on distance to the caustic as it is approached and then of a logarithmic dependence on distance behind the wave front once the caustic is passed. While nonlinear response will modify the results, they showed that very large stresses can be transmitted to the boundary of the inclusion.

Ting and Chou [T9] further employed this approach to investigate the propagation of stress gradient through an inclusion. They found that the reflection and transmission of the stress gradient and the higher-order derivatives of stress do depend on the geometries of the incident wave front and the interface boundary though it is not so for the stress. They also studied the propagation of the stress, the stress gradient and the higher-order derivatives of the stress behind the wave front. They showed that the stress gradient does not maintain the same sign as the wave front is propagated. The implication of this result is that the plastic yield can occur at places behind the wave front before it occurs at the wave front even if the initial wave has a discontinuous rise in stress at the wave front followed by a gradual decrease in magnitude behind the wave front.

An analysis based on the theory of elasticity and applicable to composites was carried out by Achenbach [A15] for welded elastic quarter-space. The case of impulsive shear load with step-function time dependence applied to the surface of the half-space formed by the joined quarter-space was considered. The shear loading was parallel to the interface of the two quarter-space so that only shear waves were generated. Because the stresses at the corners may be singular, he used a technique in which each of the waves generated at the corner was analyzed separately and then appropriately joined. The shear stress on the interface was found to have a logarithmic singularity at the surface of the half-space. This singularity of stress may cause fracture at interface corners.

Brock and Achenbach [B44] then extended the transient shear solution to the case of loads normal to the surface of the half-space formed by the two quarter-

spaces. Mixed boundary conditions were taken to be a normal velocity with zero shear stress. In this case, the shear stress at the surface is zero because of the mixed boundary condition and hence there is no singular shear stress at the corner of the joined quarter-spaces. However, they found that for step loading, a singular shear stress is propagated at the Stoneley wave velocity while the normal stresses are bounded.

Brock and Achenbach [B45] further carried out an analysis on the effect of an incident shear wave on the extension of an interface flaw between two half-spaces. Shear waves polarized parallel to the edge of the flaw and perfectly-plastic bonding were considered. They found that in case of a constant-velocity crack extension the incident stress must be a step wave. The velocity of the crack is expressed in terms of the amplitude and angle of the incident wave, the properties of the half-spaces, and the yield stress of the bond. This analysis should have an important value in interpreting fractures in composites.

Ko [K3] used a technique based on Kirchoff's method of retarded potentials to analyze the interaction of plane waves with circular inclusion. He showed that the singular stresses occur inside the cylinder which is of higher impedance than the surrounding medium. He also showed the position of caustics and associated fold-over wave fronts for inclusion wave speeds both higher and lower than those of the matrix. He then extended the analysis to the multi-inclusion case to study the successive scatterings [K4]. The study has application to fiber-reinforced composites. He showed that the wave front amplitude for the first-generation scattered wave along the center line of the row of fibers attenuates very rapidly, becoming no more than 10 percent of the initial value in the third inclusion. He made a comparison of this case with the laminated case and found much more rapid wave front attenuation in the cylindrical inclusion case. Determination of the stress behind the wave front can be, in principle, obtained by this

technique, but in practice such calculation will be extremely formidable. However, this analysis can be further extended to the case of a composite containing inclusions of arbitrary shape.

Transient wave propagation in a unidirectional composite was also analyzed by Haener, Puppo and Pagan [H2] in addition to the exact analysis for the steady-state vibrations. A circular fiber surrounded by a circular matrix shell with the outer surface confined was considered as the model. Based on elasticity theory, an approximate theory was formulated from variation of Hamilton's potentials. The resulting 4th-order differential equation was then solved for the transient state in addition to the steady state. By comparing the numerical results of the approximate theory with those of the exact theory, they found that for the dimensions considered in the composite the approximate theory is valid up to frequencies of 10^5 cycles per second. With thie approximate theory, it should be possible to consider more realistic boundary conditions and to solve more transient problems.

Drumheller [D8] made an extensive investigation of the propagation of elastic waves normal to the interface of a laminated composite. He used a computer code TIC based on the exact solution of the response in laminates to study the transient and sinusoidal wave propagation. He noted that the lowest mode has an upper cutoff frequency that tends to cause the dominant response to occur in the first mode, and he also noted that different laminates can give essentially identical first-mode behavior. In a successive investigation [D7], he developed this concept further and showed how a transient analysis of one one-dimensional laminate can be used to represent the response of a different one-dimensional laminate and very likely of materials which have the same

first-mode low-frequency behavior. Actually, he carried out an analysis
of laminate using both the original laminate properties and an equivalent
laminate which has the same first mode sinusoidal-wave behavior but which
has strikingly different densities. He showed that the linear response is
virtually identical for the two materials. Even more importantly, he
compared the nonlinear response for the two laminates by assuming bulk-
modulus nonlinearity and showed that the agreement was still quite good.
The important virture of this technique is that it can be extended to com-
plicated constitutive behavior, such as crushing and rate dependence, with
reasonable confidence and that the basic macroscopic response obtained
from geometric dispersion is properly maintained.

Riney, et al. [R9] first studied the stress-wave effects in inhomo-
geneous and porous earth materials. Then Okubo, Sve and Whitter [O1]
investigated the effect of porosity to the dispersion of an elastic step
pulse in a three-dimensional quartz phenolic composite. Subsequently,
Sve [S46] made an extensive investigation on transient elastic wave propaga-
tion in a porous laminated composite. In Sve's analysis, it is assumed that
wave propagates normal to the laminations and that the porosity is randomly
distributed throughout one constituent and is composed of small spherical
voids. The randomly distributed porosity produces Rayleigh scattering,
and also reduces the wave speed in a constituent thereby affecting the
geometric dispersion. A dissipative equation of motion is developed for
the porous material and used for a constituent of a composite. A dispersion
relation and a pulse solution are obtained to determine the significance of
porosity in a laminated composite. From his study, Sve concluded that the
Rayleigh scattering produces a small damping effect in the far-field pulse
shape and small-void porosity can be adequately simulated with an effective

wave speed. Hence, he further concluded that if the pores are small and randomly located within a constituent of the composite, it may be possible to use a continuum approach for low-frequency response calculation for pulse propagation in such a porous laminated composite.

Sve [S44] also analyzed the propagation of pulse in a dissipative laminated composite by using techniques of modal analysis and by considering the complex wave numbers. The analysis provides a far-field dispersive solution that is valid only near the head-of-the-pulse, including spatial attenuation. He modeled the effects of damping by assigning an imaginary part to the wave number and found that the oscillations about the steady value of the propagating pulse are reduced in amplitude and the rise time is increased, in comparison with the undamped case. In order to obtain the solution away from the head-of-the-pulse, it is apparent that other techniques, such as the method of stationary phase, would be required.

Chen and Gurtin [C4] discussed the propagation of one-dimensional acceleration waves in elastic and viscoelastic laminated composites by an exact analysis. The composites that they considered consist of a periodic array of alternating layers with plane boundaries. They made no assumptions regarding linearity, and assumed only that the materials have fading memory when they treated viscoelastic materials. They derived an expression for the amplitude of an acceleration wave propagating normal to the layers.

When confined to the junction points of the cells this expression has exactly the same form as that for a single (nonlinear) viscoelastic material. They used this fact to derive effective moduli for composites. In addition, they generalized their results, derived for laminates consisting of periodic cells of two layers, to cells of N layers. This theory is applicable to plane longitudinal motions in three dimensions.

Chen and Clifton [C5] discussed the transient longitudinal waves in elastic and viscoelastic bilaminates. The bilaminates consist of either elastic or viscoelastic laminates of uniform thickness and infinite lateral extent. Wave propagates in the direction perpendicular to the laminates. Using the Laplace transform technique and Floquet theory, they obtained both wave front and late-time solutions for step loading by means of asymptotic techniques. Combination of wave front and late-time solutions shows that the wave profiles consist of a rapidly decaying precursor followed by a dispersive transition to an equilibrium state. The transition region in elastic bilaminates is much smaller than for viscoelastic bilaminates.

Seymour and Mortell [S7] studied the propagation of one-dimensional longitudinal pulses and weak shocks in nonlinear elastic and viscoelastic laminated composites by an exact analysis. The composite consists of alternating laminates, with parallel plane boundaries, which repeat periodically. They assumed that the deformations undergone by the medium are of small amplitude and, in the case of the viscoelastic composite, they are of high frequency. Thus, in the analysis, simple wave solutions were superposed to deal with nonlinear waves which undergo reflection at the boundary. They showed that by an appropriate choice of the width of the laminates the nonlinear composite can appear to first order either as a linear-viscoelastic or linear-elastic material when signal is read at cell interfaces, and that at cell interfaces the deformation in a periodic-elastic composite is identical with that in an appropriate nonlinear viscoelastic material. The critical acceleration is obtained for elastic composite below which no shock forms. These results are extended to a viscoelastic composite chiefly by introducing a lumped damping coefficient which is the product of the

attenuation due to the mismatch of impedances and the rate-dependence of each layer. Finally, they analyzed the weak shock for two particular signal functions: (1) when the signal function is antisymmetric about the wavelet at which a shock forms, and (2) when the shock is at the front of the pulse.

As mentioned earlier, Barker [B6] proposed the use of a viscous-dispersion model of the Maxwell type to represent geometric dispersion in composite materials. He used his model to analyze the transient response of a layered composite medium. The result shows that the rise times are generally larger than those of the exact solution and the oscillations about the steady stresses caused by step loading are not presented in Maxwell's model solutions. It seems that the attenuation is overpredicted by this model. In fact, this technique, which is semi-empirical, averages out all the microbehavior of the composite and hence gives only a mean stress history.

A different type of "viscous" dispersion model was developed by Bade, et al. [B2]. In Bade's model a general expansion of the damping terms caused by both x and t derivatives of the velocity was postulated. The two lowest derivatives of the expansion were used to match experimental data qualitatively. They applied this viscous dispersion model to study the transient pulse propagation in a waveguide-type laminated composite medium. Comparison of dispersion model calculation with three-dimensional experimental data shows good agreement except near the region of peak pressure. They also carried out Bloch's analysis to determine the effect of the variation of properties in the propagation direction. It was found that the layering in the propagation direction has little effect on the dispersion.

Shea, Reaugh, et al. [S10] developed a theory very similar to that of Bade. They used essentially the same form as the dispersivity term of Bade, combined with a stress deviator depending upon current value of strain

and the history of strain. The model was incorporated into a computer code. A unidirectional quartz-phenolic was chosen in this program. The results of the computation obtained by imposing a step-velocity loading condition on the boundary show quite reasonable agreement with experimental data except that the detailed oscillations are not reproduced by the model.

Curran, Seaman and Austin [C13] proposed the use of artificial viscosity to compute one-dimensional wave propagation in composite materials. They proposed a macroscopic model utilizing the dispersion properties in the existing hydrocodes of WONDY and PUFF types. In such hydrocodes, the artificial viscosity was estimated. Computations were made for wave propagation in epoxy-steel laminates, quartz-epoxy, and quartz-phenolic composites for which experimental and computational data existed. The results are in good agreement with other models and experimental profiles for the main wave shapes. It is an original approach for computing large amplitude, one-dimensional wave propagation in composite materials. However, the approach is a macroscopic one in which only averaged values of stresses, displacements and particle velocities can be predicted.

Christensen [C10] discussed wave propagation in layered elastic composite on the basis of dielectric theory. Both periodic layering and random layering were considered. In case of periodic layering, the propagation became dispersive and in case of random layering, it exhibited dissipative behavior. Regarding the medium as an equivalent anisotropic one, long wave approximation for propagation normal to layering was modeled by using Botzman constitutive law. The dispersion relation was expanded for long wavelength assumption and compared with known results to determine the unknowns in the relaxation function. Results were compared with known experimental data and very good agreement was observed. The theory was

applied to study the problem of a pressure pulse acting on a half-space of the layered medium. The response was found to be identical with that obtained by Hegemier [H8]. The theory proposed may be regarded as an effective modulus model with viscoelastic constitutive law.

Drumheller and Sutherland [D10] proposed a lattice model to study the transient waves in composite materials. Based on the assumption that geometric dispersion results mainly from the relatively periodic arrangement of the reinforcing elements in the matrix rather than from the precise shape of each reinforcing element, they developed a lattice which ignores the shape of the reinforcing elements but preserves their periodicity. In the application of this lattice model, the composite was initially treated as a nondispersive homogeneous mixture. The effective or average properties of the mixture were determined by either steady-wave analysis or appropriate experiments. A lattice was then formed by redistributing the mass within the mixture to form a periodic structure of laminated plates. This mass redistribution was carried out in a manner which yielded a lattice with theoretical dispersive characteristics that matched the measured dispersive characteristics of the composite. The model was applied to composites, consisting of a regular array of tungsten fibers in an aluminum matrix, subjected to a step loading. Also, flyer-type impact experiments were performed in the plastic range of the composites. The agreement between experiment and calculation for the arrival time and rise time of the wave front and for the frequency of the ringing behind the wave front is good. It seems that for a wide range of engineering applications, this model can be used to predict the behavior of actual engineering composites.

Nayfeh [N6] presented a discrete viscous lattice model to simulate transient motions in elastic and viscoelastic composite. Viscosity was

introduced via dashpot in the lattice. Integral transform techniques were used to solve the lattice problem. From the numerical results, he found that discretization in the model introduces oscillation about the continuum solution, and viscosity damps those oscillations.

IV.   EXPERIMENTAL INVESTIGATIONS

Experimental work on the propagation of waves in directionally-reinforced composites has been carried out by many investigators by means of ultrasonic techniques; pulse transmission measurements, and other techniques.   In this sections, summaries of the experimental investigations are given.

IV.1 Ultrasonic Techniques

Asay, et al. [A24] measured sinusoidal-wave dispersion using ultrasonic techniques developed by Asay for homogeneous materials.   They tested carbon-phenolic laminates reinforced with layers of high-modulus filaments spaced about 0.6 mm apart.   The results showed a pronounced variation of the phase velocity with the frequency.

Tauchert and Guzelsu [T3] investigated the dispersion behavior of plane-harmonic waves in a boron-epoxy composite using ultrasonic techniques.   They determined the dependence of group velocity upon the frequency for longitudinal and transverse waves propagating either parallel or perpendicular to the fibers.   It was found that transverse waves propagating in the direction of the fibers show a very pronounced dispersive behavior, and that the group velocity increases with the wave number.   This type of behavior is consistent with that predicted by the continuum theories.

Bedford, Sutherland and Linge [B14] carried out extensive investigations on a fiber-reinforced composite of tungsten wires unidirectionally embedded in a 6061 aluminum-alloy matrix.   The composite chosen for the experiments was prepared for two constituent ratios, 2.2 and 22.1 percent by volume of tungsten, respectively.   Ultrasonic experiments were conducted for plane compression waves propagating normal to the direction

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of fibers by using water-bath techniques with wide-band transducers. Experimental dispersion data for the two lowest modes agree well with the predictions based on continuum theory of mixture, except at the highest frequencies. The dispersion data obtained demonstrate that for propagation normal to the direction of the fibers, fiber-reinforced composites behave as wave filters which selectively transmit or reflect periodic waves.

Robinson and Leppelmeir [R10] experimentally studied normal propagation of ultrasonic shear waves in a layered steel-copper composite. Samples were manufactured by diffusion bonding alternate layers of steel and copper foil together. The passing bands and stopping bands in the frequency curve were observed. The experimental dispersion fits existing theory [S40,K5] quite well.

Rose and Mortimer [R11] measured wave velocities in unidirectional graphite-epoxy shells with four fiber orientations. Ultrasonic techniques and drop mass impact were both used to generate longitudinal pulse. Good agreement with anisotropic shell theory was found.

Reynolds and Wilkinson [R8] studied experimentally the ultrasonic waves in CFRP (carbon fiber-reinforced plastics) composites. Experiments were conducted on two- and three-ply laminates having various orientation and ordering of the layers. Results were presented in terms of shear and compression wave velocities versus propagation angle to the fiber direction of uniaxial materials. Experimental data agree well with theoretical predictions. In addition, resin porosity was found to have important influences

on the propagation - wave velocities decrease with increase of percentage of resin porosity.

Felix [F1] determined experimentally phase velocity and attenuation for several plastics. The results for low amplitude were represented adequately with a standard linear viscoelastic model within the limited frequency range. He emphasized that, for composites using the plastics as matrix, material attenuation as well as geometric dispersion must be considered in analysis of pulse propagation. Reliable prediction should be obtained by using the measured phase velocity of the medium together with a frequency-dependent attenuation for stress pulse propagation in a composite material.

Chang, Couchman and Yee [C2] conducted ultrasonic resonance measurements of sound velocity in thin composite laminates. This could be an important addition to the work on an experimentally difficult problem. Measurements were made by an ultrasonic pulse-echo technique. However, the description of the experiments is somewhat sketchy. It would be difficult to duplicate the experiments with the information provided.

Rose, Wang and Deska [R12] experimentally determined the wave surface in a unidirectional graphite-epoxy plate by using the ultrasonic technique. The results are in agreement with the composite wave propagation theory proposed earlier by Yang, et al. [Y2] and extended by Wang and Tuckmantel , [W4]. Recently, Martin [M15] presented a new method for the measurement of phase velocity of ultrasonic waves in elastic composites.

IV.2 Flexural Resonance Techniques

Schultz and Tsai [S2] obtained experimental data on moduli and damping ratios of fiber-reinforced composites by studying the free and forced transverse vibrations of cantilever beams made of the materials. The composite

exhibits anisotropic, linear viscoelastic behavior when undergoing small oscillations. The data obtained are useful to designers concerned with vibrations and impact loading of filament-reinforced composite structures.

Using vibrating cantilever beam specimen, Tauchert and Moon [T1] performed experiments to determine the dependence on frequency of the complex moduli of unidirectional glass-epoxy and boron-epoxy materials. Considerable data were presented. The damping tends to increase with frequency whereas the modulus is relatively constant. At high frequencies, where shear deformation and rotary inertia are significant, the specimens are characterized as Timoshenko beams. Using these data, they compared predicted and measured velocity and attenuation of longitudinal pulses in the rods. They concluded that the linear theory of viscoelasticity is adequate for predicting velocity and attenuation of longitudinal pulses in composite rods of unidirectional glass-epoxy and boron-epoxy materials over a large frequency range.

Tauchert [T2] also obtained flexural resonance data for a class of woven-fabric composites. The data were used to measure the complex moduli, and the results, extrapolated to some degree to extend into the frequency range pertinent to stress wave propagation, were then applied to velocity and attenuation prediction. However, he did not discuss dispersion and scattering phenomena which may have introduced sharp changes in the complex moduli.

IV.3 Shock Tube Tests

Whittier and Peck [W15] tested impulsively a laminate of carbon-phenolic layers reinforced with layers of high-modulus filaments in a shock tube. A step-pressure rise was applied to one side of the specimen by reflection of

a gas dynamic shock wave. The spatial average of the free-surface velocity over the cross section of the composite was measured with a capacitance gage. The experimental results agree well with the head-of-the pulse approximation obtained by Peck and Gurtman [P3]. The shock tube test was found to be an effective method to characterize the dispersivity of a composite in avoiding the complications of nonlinearity.

Sve and Okubo [S48] investigated the rear surface velocity of three laminated composites with lamination angles of 0°, 45°, 90° to the loading direction by using capacitance transducers. A step pressure pulse was imparted to the specimens using a shock tube, with response confined to the linear elastic range. Experimental results confirm the conclusions presented in Sve and Whittier's [S41] theoretical studies which are: (i) elastic response is composed of two contributions that travel at different speeds; (ii) the dispersion is also different for each of the contributions and depends on the low-frequency behavior of the phase velocities; (iii) far-field average responses of 0° and 90° lamination composites are similar, while near-field responses of them are not similar.

IV.4   Impact Tests

Washington [W6] made near-field measurement of stress waves propagating parallel to Plexiglas-aluminum laminates. The laminates consisted of aluminum plates alternating with Plexiglas plates. The stresses were generated by aluminum or Plexiglas-flyer plate, with thicknesses such as to produce load durations of 0.6 to 0.8 μsec. The stresses were measured at several depths, using Manganin wires embedded in epoxy, located primarily at the center of each laminate. The results showed strong attenuation in the aluminum layers, even for 0.8 μsec. load, but little or no attenuation in the Plexiglas. It seems that at these short propagation distances one is

observing, primarily, the transfer of energy from the aluminum into the Plexiglas layer, as opposed to overall pulse attenuation.

Seaman, et al. [S6] investigated experimentally the behavior of three-dimensional orthogonal quartz-phenolic composites at moderate to high stresses. Five plate-impact shots were made, four with thick flyers for obtaining Hugoniot data, and one with a thin flyer for pulse attenuation data. The response was measured with Manganin wire gages embedded in epoxy behind the specimen. Hugoniot data derived from the flyer velocity and shock velocity were closely fitted by a straight-line in shock velocity versus particle velocity, and fair agreement of these data with the gage stresses was also obtained. The rise-time data suggest that the pulse-spreading effects of geometric dispersivity were overridden by the steeping effects of nonlinearity as the peak stress was increased. The pulse attenuation results indicate both the highly attenuative nature of the material and the inadequacy of those predictions which do not account for geometric dispersion.

Calvit and Watson [C1], and Sutherland and Calvit [S38] performed a series of dynamic experiments for fiber-reinforced viscoelastic composites subjected to uniaxial pulse. The composites they tested were viscoelastic resins reinforced with (i) neoprene filaments, (ii) roving glass fibers, and (iii) nylon 66 fibers. The transient pulse was initiated by a mechanical striker and the particle velocity was measured by using the so-called Faraday transducer. The particle velocity records were converted into phase velocities and attenuation coefficients. Experimental results were compared with the predictions of effective modulus theory for the three kinds of reinforcement and with the predictions of Bedford and Stern's mixture theory for the latter two kinds of reinforcement. They concluded

that: (i) if the effects of internal geometry are small, the effective modulus theory may be used to predict the dynamic response of a fiber-reinforced material; if the effects are not small, the prediction of the mixture theory is more accurate; and (ii) for a constant volume ratio of fiber material, the deviation of the theoretical model from the experimental results increases as the number of fibers per unit area decreases.

In a series of papers, Lundergan and Drumheller [L8,9,10,11] investigated the dispersion of dilatational stress waves in a laminated composite both experimentally and analytically. The composite consisted of a number of bilaminar plates, and the propagation of the wave was normal to the plates. A flat flyer-plate accelerated by a compressed gas gun was used to induce a rectangular stress pulse. The particle velocity was measured at the opposite end of the composite by an optical interferometer. The experimental results were compared with results obtained by means of computer programs in which the layers are explicitly represented, and which include nonlinear effects. In the first set of experiments [L8], layers of epoxy and steel were impacted at stress levels on the order of a few kilobars. It was found that the peak amplitude of the transmitted stress wave decreased directly with the width of the input stress pulse. The bulk of the reduction of the stress was attributed to the reflections of the stress wave at the extreme left and right boundaries of the composite. Debonding of the bilaminate plates and dissipation of energy in the epoxy material were also thought to have contributed to the stress reduction. Comparisons of the experimental results with numerical results provided an indication of the necessity of considering the nonlinear behavior of the materials.

In subsequent experiments [L9] laminates of PMMA and stainless steel were used. By varying the number of laminates it was concluded that there was no change in rise time for propagation distances beyond a few unit cells from the loading face. Another set of experiments [L10] was carried out on the epoxy-steel laminates to investigate the response at slightly higher stress where the effect of fracture would be more apparent. It was found that the fracture location depended on the pulse duration, with fair agreement between the predicted and experimentally determined fracture locations. Subsequently, Lundergan and Drumheller [L11] investigated the response of obliquely-laminated composite, both experimentally and analytically. Several analytical models and a two-dimensional wave propagation program were used to predict the transmitted wave form. Comparisons were made between the various models and the experimental results. General agreements exist between the predictions of various models. The experimentally determined first-signal velocities and final wave amplitudes agreed with the calculations of the models; however, the remaining portion of the experimentally determined stress waves exhibited slower rise times than did those of the calculated waves. It was speculated that the physical conditions existing at the input boundary of the composite were not being adequately incorporated into the models.

Berkowitz and Cohen [B33] studied experimentally high amplitude stress-wave propagation in an anisotropic quartz-phenolic composite. They performed Hugoniot and pulse-attenuation plate-slap tests on quartz-phenolic laminates. They tested composites with lamination planes normal to and parallel to the propagation direction. They concluded that the composites exhibited rate-independent plasticity. Then they used a plasticity model and were able to let the attenuation match the experiments with some discrepancies in pulse

shape. The number of tests was too few to validate the model conclusively, but the validity of the plastic model was demonstrated.

Reed [R4], and Munson, Reed and Lundergan [M14] performed thin pulse attenuation experiments on a cloth-laminate quartz-phenolic composite. Thin-pulse tests for propagation normal to the layers indicated that: (i) much more attenuation occurred than would be predicted by simply using a hydrodynamical model for quartz-phenolic; (ii) an empirical fit to the Barker [B6] viscous-dispersion model relaxation time gave very good agreement with the attenuation curves. Measurements on a three-dimensional quartz-phenolic composite with fairly high porosity showed that the same Barker model did not predict as large attenuation as was found experimentally. It remained a matter of controversy, in three-dimensional quartz-phenolic composite tests, as to whether the increase in attenuation in the three-dimensional material was caused by the crushing, by the geometric dispersion, or by a combination of the two effects. Other experiments indicated good comparisons with both micromechanical models and continuum theories of Bade type [B2], without accounting for porosity.

Reed and Munson [R5] again investigated stress pulse attenuation in cloth-laminate quartz-phenolic both analytically and experimentally. The composite was modeled as a homogeneous viscoelastic material (Maxwell type) with nonlinear storage modulus and one relaxation time. Judicious curve fitting from one plate-impact test produced the time constant, and subsequent comparisons of analytical predictions with experimental results were based on this value. Experimental results showed marked dispersive spreading of the wave profiles and very strong attenuation of stress, but no evidence of elastic-plastic effects. Model predictions agreed very well with the measured profiles. Attenuation prediction was also quite good. The data

and analysis suggested that the viscoelastic nature of the phenolic matrix plays a dominant role in the attenuation behavior.

Michael, Christman and Isbell [M3] studied experimentally the pulse propagation in composite materials. The pulse was generated by flyer-plate impact and particle velocity was measured by using velocity gage. The materials tested were COMRAD 1 composite and laminated quartz-phenolic. Measurements on COMRAD 1, with pulse propagation normal to fibers, showed that the structure of the pulses was approximately the same for linear and nonlinear response levels. Tests on laminated quartz-phenolic indicated that the response to step loads was somewhat dispersive. It appeared that rate dependence and/or plasticity played a role as large or larger than that played by the geometric disperion.

Holmes and Tsou [H18] investigated the steady shock waves in fiber-reinforced composite materials. The shock wave was generated by a planar impact of a flyer plate and propagated along the direction of the fibers. The sample used in experiments was made of unidirectional aluminum fibers cast in an epoxy matrix. Both shock-wave velocity and free-surface velocity were measured by means of optical techniques. The shock front in the composite was found to be steady. The results of the measured Hugoniot also gave a satisfactory comparison with those obtained from an analysis proposed earlier by Tsou and Chou [T13]. The experimental justification of steady shock front is quite significant since the assumption of steady shock can be utilized to solve problems involving composites of various configurations.

Barker, Lundergan, Chen and Gurtin [B8] studied experimentally the shock waves and acceleration waves in laminated composites. To verify the prediction of Barker's model [B6], they considered a composite consisting of alternating layers of polymethyl-methacrylate and aluminum. Not only did

the experiments establish the existence of shock waves in low stress region, but the nonlinear viscoelastic model predicted quite accurately their gross structure. Also, to verify Chen and Gurtin's results [C4], they considered a composite of alternating layers of polymethyl-methacrylate and cross-cut quartz. They generated in this composite acceleration waves with input amplitudes both above and below the critical amplitude a_c, and in both of these instances the results predicted by the theory were verified. These experiments were the first to produce, in composites, steady shock waves at such low stresses, and the first to display acceleration waves in composite materials.

IV.5 Fracture Tests

Warnica and Charest [W5] tested by means of plate impact laminated quartz-phenolic with the layers inclined at 90°, 0°, 45° to the direction of wave propagation. Pulse durations were 1 to 2 μsec. A definite spall threshold was obtained only for the 0° composites, where spallation occurred due to delamination. In the cases of 45° and 90° angle inclinations, delamination occurred before spallation. In the 0° case, a significant difference in stress was observed between the onset of microscopic cracking and macroscopic spall.

Green, Babcock and Perkins [G9] investigated experimentally the problem of degradation of mechanical properties of composites. They found that laminated quartz-phenolic composite tested with momentum traps up to 9 k-bar had no degradation in compressive properties. For carbon-phenolic laminates, stress up to 7 k-bar had no effect but at 8 k-bar a strength decrease of 20-40% was found. In these experiments, it was felt that not all extraneous damage was suppressed and the specimens might have felt some tension as well as compression.

Perkins, Babcock, Shierlock and Jones [P5] conducted an experimental study of the effect of impulsive preloadings on the behavior of quartz-phenolic composite. Quartz-phenolic laminates were impact-tested with momentum traps. For compression strength and stress wave loading parallel to the lamination, degradation occurred at both room and elevated temperatures. For tension strength and stress wave loading normal to the lamination, degradation occurred at room temperature. Uniaxial strain and uniaxial stress preloadings were performed on quartz-phenolic to assess the effect of an impulsive preload on the subsequent uniaxial stress behavior of a composite material. In these tests observable damage was limited to the phenolic matrix. As a result, post-preload experiments on the specimen under uniaxial stress correlated the dependence of the fracture strength on the extent of the damage to the phenolic matrix as well as the direction of the preload relative to the fiber lay-up.

Schuster and Reed [S3] investigated the fracture behavior of boron-aluminum composite materials by means of shock loading. The composite was formed from boron filaments and aluminum matrix. The boron filaments ran perpendicular to the direction of stress-wave loading in two orthogonal directions in the plane of the specimen. Two types of aluminum matrix were used: one was brazed and the other was diffusion-bonded to the filaments. Both composites suffered filament cracking thought to result from the compressive wave passage. The spall cracks formed on plane parallel to the free surface of the composite.

Reed and Schuster [R6] further studied the filament fracture and post-impact strength of boron-aluminum composites. They tested specimens with momentum traps to suppress spall and then measured tensile strength in the plane of the specimen. They found the degradation began to occur at a fairly

well-defined flyer velocity higher than that required to spall aluminum.
Retained strength as low as one third of the normal strength was obtained
at the highest flyer velocities. It appeared that the primary mode of
damage in these specimens was the debonding of the fibers from the surround-
ing matrix. The investigation separated the effects of compressive shock
loading from the effects of spallation. Fracture of the filaments was shown
to occur during the initial passage of the loading wave.

Barbee, Seaman and Crewdson [B5] made an experimental study of dynamic
fracture of a quartz-phenolic composite and a silica-phenolic composite.
The quartz-phenolic composite composed of unidirectional closely-packed
yarns was tested with wave propagation parallel to the yarns. They found
that cracks both parallel and perpendicular to the yarns occurred in the
matrix. The composite was also tested with momentum traps and then cracks
were found only parallel to the fibers. Crack was compressive in origin.
For a laminated silica-phenolic composite, no compressive damage occurred
when tested with momentum traps, but it was found that when tested without
momentum traps a spall line was formed roughly parallel to the free surface.
The spall took place partly by debonding between the layers and partly by
fracture though the layers, as was required to form a spall plane parallel
to the free surface.

Gerberich [G3] experimentally analyzed the fracture behavior of a com-
posite with ductile fibers. Aluminum matrix composites reinforced with
unidirectional stainless-steel wires of 450 ksi tensile strength were
analyzed in terms of strength, stiffness, and fracture characteristics
along and across the fiber array. Tensile strengths as high as 200 ksi were
obtained in the fiber direction while 150 ksi transverse strengths were re-
corded. Longitudinal strength and moduli agreed reasonably well with rule
of mixture predictions up to 40% fiber by volume. Interfacial bonding was

sufficient to provide transverse stiffness enhancement but did not contribute to transverse strength. Crack propagation across fibers was found to be controlled by the very ductile high strength fibers. Crack propagation between fibers was controlled by fiber spacing. Critical stress intensities for transverse crack propagation were reported to be as low as one third that of the matrix.

Sierakowski, et al. [S12] presented results of an experimental program systematically evaluating the deformation and fracture of steel wire-reinforced epoxy composite systems. The program involved mechanical testing in the strain rate range 10^{-5} to 10^{3}/sec. and impact testing using massive elastic targets at strain rate approximately 10^{4}/sec. Specific results included static and dynamic properties, strain rate sensitivity, information on the nature and character of dynamic fracture, influence of specimen geometry and reinforcement spacing, etc. Further, they proposed a simplified energy criterion for predicting failure modes and critical velocities for composite specimens with a brittle matrix.

Cohen and Berkowitz [C12] also studied experimentally dynamic fracture of a quartz-phenolic composite under stress-wave loading in uniaxial strain. They carried out thin-flyer tests using 5 mil and 15 mil Mylar flyers impacting 0.25 inch thick composites. They found significant difference between microscopic and gross spall, and that spall occurred by delamination. They found secondary cracks perpendicular to the impact faces which were most intense near the front and rear faces of the material and could have been caused by compression wave or late-time flexure of the specimens. They also found that a constant stress tensile fracture criteria applied in a low impulse region while a rate process criteria applied in a high impulse region.

Hoover and Guess [H19] designed experiments to measure the dynamic fracture toughness K_D and the work-of-fracture parameter γ_{fD} as functions of the rate of loading. K_D is the critical stress intensity factor at which crack initiation occurs and γ_{fD} is a measure of the energy absorbed as a crack initiates and propagates through the material. The materials tested were carbon-carbon composites having almost exclusive application in space industry. The data obtained should be of practical use, and the technique described may be useful for the dynamic testing of other materials.

Drumheller [D9] investigated the effect of debonding on stress wave propagation in composite materials both analytically and experimentally. Differing from Sve's approach [S42] by introducing debonding, he studied the phase velocity behavior in the limit of zero wave number. He found that in the limit there are three values of phase velocity instead of two which were found by Sve without accounting the effect of debonding. He confirmed these three values by experiments on a laminated composite consisting of stacks of stainless steel and polymethyl-methacrylate plates. To obtain good agreement between theory and experiment for the third velocity, it was found necessary to employ high-pressure polymethyl-methacrylate data.

Drumheller and Norwood [D13], and Drumheller and Lundergan [D14] further studied the debonding effect on the behavior of stress waves in composite materials theoretically and experimentally. The theoretical investigation was focused on the problem of a debonded composite, a situation commonly observed in flyer-plate impact experiments. In contrast to the fully bonded case, an additional stable wave mode, they suggested, may propagate in the composite. Thus, another boundary condition is required to obtain the solution. They postulated such a condition. The essential concept was to introduce an interface warping stiffness relating the difference

between the constituent stress and the average interface stress, with the
difference between the corresponding displacements. The theory was applied
to a configuration modeling the flyer-plate experiment. Experiments were
then conducted on a composite formed from laminates of 6061-T6 aluminum and
polymethyl-methacrylate. Pressure were sufficiently high to cause debonding.
Some difficulties arose in comparing theory and experiments. The theory
was based on the assumption of total debonding. However, the experimental
results indicated significant bond strength. After they modified the
theoretical results by an empirical correction to calculated wave speed,
quite reasonable agreement was obtained. The theory presented is unsatis-
factory in several respects as they admitted.

Baldwin and Sierakowski [B4] investigated the uniaxial static and
dynamic fracture characteristics of a composite material consisting of an
aluminum matrix, A-13 casting alloy, and stainless steel fibers, type 304.
Dynamic compression specimens loaded parallel to the fiber direction failed
by filament buckling, while loading transverse to the filaments produced
fiber matrix debonding. The composite system tested did not exhibit any
rate-sensitivity in its failure characteristics.

V. COMPARISONS AND RECOMMENDATIONS

V.1 Comparisons of Various Theories

The effectiveness and the applicability of a theory depend on the kind of solution one is looking for. A theory may be very effective in finding the solution to some aspects of wave propagation but is totally unsuitable for finding the solution to other aspects of the wave phenomena. The applicability of a theory, in most instances, reflects the assumptions made in the theory. In comparing the applicability of various theories, we will discuss some of the better established theories as to what limitations and assumptions are imposed on the theory, and the merits and drawbacks of the theory. In particular, we will look at the following questions: Is the theory applicable to nonlinear composites and non-periodic composites? Can the theory be easily used for solving the transient problems? How good is the approximation and how difficult is it to include the higher order terms to get a better approximation?

V.1.1 Effective modulus theories

In the effective modulus theories, a composite is replaced by an anisotropic linear solid whose static responses are equivalent to the macroscopic responses of the composite. The relations between the macroscopic stresses and strains are derived solely on the bases of static loading. With the effective modulus theories, the wave speeds in a composite are independent of the frequencies. Therefore, the dispersion phenomena observed in the experiments are not predicted by the effective modulus theories. Consequently, the theories are totally inadequate for predicting the dynamic responses of a composite.

For the static response of a composite, the effective modulus theories are quite adequate. The theories can be, and have been, applied to nonlinear

composite materials such as elastic-plastic composites. There is no reason why the theories cannot be extended to non-periodic composites, although the analyses would become complicated. In most effective modulus theories, the ratio of the micro-dimension to the macro-dimension is not critical. The theories are exact regardless of the ratio.

V.1.2 Effective stiffness theories

There are two crucial steps in the effective stiffness theories. We will use the bilaminates as an example. Firstly, the displacements within each layer is expressed in a Taylor series (or other polynomials) about the midplane of the layer. The coefficients of the series are therefore defined only at the discrete points, the midplanes. The continuity of displacement at the layer interface yields a finite difference equation. With the assumed displacement field, one obtains the strains, strain energy and kinetic energy in each layer. The second important step is the smoothing operation in which the functions previously defined only on discrete points are extended to defined for all points. This is accomplished by taking a weighted average of the strain energy in the layers. Assuming the smallness of the layer thickness, the continuity condition is rewritten in a differential form and hence a continuum theory is developed.

Application of Hamilton's principle is to obtain the best approximate solution for the assumed displacement field. This results in a system of differential equations for the displacement in the composite.

The accuracy of the theory depends on the series expansion of the displacement in the layer which in turn depends on the thickness of the layer. Higher order approximation can be obtained but one has to do the entire derivation from the very beginning. Application to transient problems is a

matter of solving the system of differential equations and there are 12 of
them for the lowest order of approximation. Since the basic elements in
the theory are the assumption of the displacement field, smoothing operation,
and application of Hamilton's principle, the theory can be, in principle,
extended to nonlinear system. However, solving 12 nonlinear differential
equations is not a simple matter.

Application of the theory to non-periodic composites is not possible.
However, the theory can be applied to periodically layered composites in
which each unit cell consists of more than two dissimilar layers.

V.1.3 Theory of interacting continua

In the theory of interacting continua, not only the displacement, but
the stress also are expanded in a power series in space variables about the
midplane of each layer. Instead of using the Hamilton's principle to obtain
approximate differential equations for the coefficients of the power series,
the entire power series for the displacement and the stress are substituted '
into the equation of motion and the constitutive equation to obtain differen-
tial-recurrence relations for the coefficients of the power series. Therefore,
coefficients of the higher order power are expressed in terms of the coeffi-
cients of the lowest order power in the stress and the displacement. The
original power series expansions are rewritten in series of time differen-
tiation of the coefficients of the lowest order expansions. Application of
the continuity in stress and displacement at the layer interfaces yields
four differential-difference equations for the stress and displacement
at the midplanes of the two-layered composite. The finite difference
in the differential-difference equation contains the thickness of a
unit cell in the composite. The assumption of smallness of

the thickness allows one to expand the difference equation in Taylor series and one has four partial-differential equations of infinite order.

As it turns out, each differential operator is accompanied by the parameter ε which is the ratio of the thickness of a unit cell to the typical macro-dimension ℓ. Moreover, the differential operators are all of even orders. The zero order approximation yields the effective modulus theory. The first order theory (which is first order in ε^2 and could have been called second order) yields a result somewhat better than the effective stiffness theory. This is not surprising since the first order (ε^2) theory takes into account the continuity in displacement and stress at the layer interface whereas the first order theory in the effective stiffness theory takes into account the continuity in displacement only. Presumably, one could develop a second order effective stiffness theory which will be equivalent to the first order theory of interacting continua.

The theory of interacting continua has many useful features. Firstly, the improvement of the accuracy by including the higher order terms is easier to accomplish. For the bilaminates, in particular, we have an explicit closed-form expression for any order of accuracy we wish to obtain by using Eq. (32) derived on p. 13. Secondly, this theory can be used either for a steady state vibration of the composite or for a transient response of the composite. Thirdly, at least for the bilaminates, the exact frequency equation is readily recovered from the theory.

There is no reason why the theory cannot be extended to nonlinear composites. However, it would certainly become unwieldy if an order of higher than the first one is attempted. As in the effective stiffness theory, application to non-periodic composites is not possible.

V.1.4 Mixture theories

In the mixture theories, the constituents are assumed to coexist and are allowed to have an independent motion, even though perfect bonding between the constituents is understood. The assumption on the relative motion of the constituents is the crux of the theories and is also the main roadblock for improving the accuracy of the theories. Although satisfactory results are obtained for harmonic waves in a simple bilaminate, the problem of assuming a suitable interaction between the laminates still remains.

Mixture theories can be applied to both harmonic waves and transient waves. Extension to nonlinear composites is possible but again the difficulty lies in defining the interaction between the constituents.

V.1.5 Other theories

There are other less widely used theories which we will not compare in detail here. However, a few words about the Floquet theory and the variational techniques are in order.

Floquet theory applies to a system of linear differential equation with coefficients which are periodic functions. Therefore, it can be readily applied to composites whose constituents are arranged periodically. The most one can get from applying the Floquet theory to a composite is the form of a steady wave train propagating in the composite and the relation between the frequency and the wave number of the wave train. For a transient wave, one has to superimpose wave trains of all frequencies to achieve the specified initial and boundary conditions. Therefore, Floquet theory is most convenient for solving steady state motion in linear, periodic composites. It is not suitable for transient wave motions except when only an asymptotic solution for large time is desired.

In variational techniques, the governing differential equations for wave motions in a composite are written in a form of integrals such that, by setting the first variation of the integrals to zero, one recovers the governing equations. The form of the integrals is not unique, and hence some variational methods are more effective than others. Unlike the Floquet theory, variational methods are not limited to linear systems, although in literatures they are applied mostly to linear composites. Even for linear composites, variational techniques are used for solving steady-state vibrations only. Applications to transient problems are possible theoretically but impractical. Some of the difficulties in applying the variational principles are the assumption of the test functions and how one chooses the next improved test functions. Therefore, the choice of test functions requires subjective judgements except in simple cases where the judgements are trivial.

V.1.6 Concluding remarks

Although various theories are available for treating linear, periodic composites, they can be equivalent to one another if a proper assumption is made in each theory. An example was given by Hegemier [H1] in which he showed the equivalence of the modified first order theory of interacting continua to the binary mixture theory for waves propagating normal to the layering of a composite. By combining the equations for the momentum and the constitutive relation for a layered composite, he obtained the constitutive equations in binary mixture form in which the "interaction" term for the mixture theory was deduced.

One can also compare various theories by looking at the frequency equations of each theory. It should be noted, however, that if a theory

is a first order theory, one can only look at terms which are of first order. It does not make sense to include the higher order terms for comparison when the theories are of first order. For instance, for harmonic waves propagating normal to the layering of a composite, the first order frequency equation based on the theory of interacting continua can be obtained from Eq. (26) on p. 12 as

$$(1 - a_2 \epsilon^2 k^2) - (1 - b_2 \epsilon^2 k^2 c_p^2) c_p^2 = 0 \tag{59}$$

where c_p, k denote non-dimensional phase velocity and wave number, respectively, and ϵ, a_2, b_2 are defined in Eqs. (23) and (27). Equation (59) provides c_p^2 as a function of $\epsilon^2 k^2$. If we assume that

$$c_p^2 = 1 - A \epsilon^2 k^2 + O(\epsilon^4) \tag{60}$$

it can be shown that, by substituting Eq. (60) into (59) and equating the coefficients of same powers in ϵk,

$$A = a_2 - b_2 \tag{61}$$

$$= \frac{1}{5} \left\{ 1 - 2\left(\gamma^{(1)^2} + \gamma^{(2)^2}\right) + \left(\gamma^{(1)^2} - \gamma^{(2)^2}\right)^2 \right\}$$

where use has been made of Eq. (27). To compare Eq. (60) with the frequency equation based on effective stiffness theory [S18], we rewrite Eq. (82) of [S18] in the following form by using the non-dimensional notations employed here

$$\left\{ \frac{1}{16} \gamma^{(1)^2} \gamma^{(2)^2} \epsilon^4 k^4 + 3\left(2 + \frac{\gamma^{(2)}}{\gamma^{(1)}} \kappa + \frac{\gamma^{(1)}}{\gamma^{(2)}} \frac{1}{\kappa}\right) \gamma^{(1)^2} \gamma^{(2)^2} \epsilon^2 k^2 \right\} c_p^4$$

$$- \left\{ 3\left(\gamma^{(1)^2} + \gamma^{(2)^2}\right) \epsilon^2 k^2 + 144 \right\} c_p^2 + 144 = 0 \tag{62}$$

In Eq. (62), the matrix layer and the reinforcing layer are denoted by superscripts (1) and (2), respectively, and κ is defined by Eq. (23). Again, assuming that c_p^2 is given by Eq. (60), one obtains

$$A = \frac{1}{48} \left\{ \gamma^{(1)^2} + \gamma^{(2)^2} - \left(2 + \frac{\gamma^{(2)}}{\gamma^{(1)}} \kappa + \frac{\gamma^{(1)}}{\gamma^{(2)}} \frac{1}{\kappa} \right) \gamma^{(1)^2} \gamma^{(2)^2} \right\} \tag{63}$$

Since this is not the same as Eq. (61), the first order theory of interacting continua and that of effective stiffness are not equivalent. However, one could make the two theories equivalent if, in the effective stiffness theory, a different assumption than that of [S18] is made on the smoothing operation in which the weighted average of strain energy and kinetic energy were calculated. One could also make a different assumption on the displacement field in the layers.

A remark on higher order approximation should also be in order. For the theory of interacting continua, the frequency equation for higher order approximation is obtained by adding terms of $(\epsilon k)^4$, $(\epsilon k)^6$,.. in Eq. (60) without changing the coefficient of the first order term given by Eq. (61). For the effective stiffness theory, a second order approximation studied in [D12] appeared to have changed the coefficient A given by Eq. (63) to that given by Eq. (61) because the frequency curve of the second order appeared to have the same curvature as that of the exact solution at $k = 0$.

In summary, it is seen that the differences in the theories are (a) the assumptions made, (b) the ease in improving the accuracy, (c) the ease in solving transient problems and (d) extensions to nonlinear and non-periodic composites. The theory of interacting continua seems to be the best among all theories when all merits and drawbacks are taken into consideration. This, of course, does not mean that one should use the theory of interacting continua for all problems.

V.2 Recommendations

V.2.1 Linear composites

For the response of composites whose governing differential equations are linear, the various theories presented here are adequate for predicting

the first mode frequency equation. For the transient problems, the theories are also adequate for predicting the head-of-the-pulse propagation at large time and at large distance from the point of impact. The theories are in general less successful in predicting, say, the space-wise stress variation at any fixed time. In particular, the stress near the impact end and at points which are at finite distance from the impact end has not been predicted satisfactorily by any of the existing theories. It is felt that a theory based on viscoelastic analogy between an elastic composite and a viscoelastic solid may be able to accomplish this objective. The approach may predict not only the short-time response but also the long-time head-of-the-pulse response.

V.2.2 Nonlinear composites

Although some attempts have been made to analyze the dynamic response of nonlinear composites, there seems to be no theory available which can predict the transient response of nonlinear composites satisfactorily. Unlike for linear composites, improving the accuracy of an existing theory for nonlinear composites presents another difficult problem. The difficulty of analyzing the transient response of a nonlinear composite can be illustrated by considering wave propagation normal to the layering of a bilaminate due to a step normal load on the surface of the composite. For this simple one-dimensional problem, the waves in the first layer after the application of the normal load will be simple waves or a shock wave depending on the stress-strain relation of the layer. Assuming that the waves are simple waves, the reflected waves from the interface boundary between the first and the second layer will be an unloading wave which may generate a shock wave. The waves transmitted to the second layer will be

simple waves. As one can see, the solution is already complicated enough after the first reflection and transmission. For a composite which contains many layers and hence will have multiple reflections and transmissions, any attempt to solve the problem exactly or nearly exactly is not likely to succeed.

What approaches one should use to solve nonlinear composite problems require further investigations. It seems that the first problem one should study is the one-dimensional waves propagating normal to the layerings of a nonlinear elastic bilaminate. Clearly, the shock waves within each layer have to be smoothed out so that only the macroscopic response is obtained. If elastic-plastic materials are used in the layerings, one has the additional difficulty of tracing the unloading and reloading boundaries. As in the case of linear composites, a theory based on nonlinear viscoelastic modeling of nonlinear composites may prove to be the most practical approach for solving transient wave problems in nonlinear composites.

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